Bulletin of the Iranian Mathematical Society
Vol. 24, No. 2, pp 1-13 (1998)

EXISTENCE OF UNBOUNDED QUADRATURE DOMAINS FOR THE P-LAPLACE OPERATOR

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In this paper we extend the notion of the quadrature domain with respect to the p-Laplace operator, for $1 < p < +\infty$. We will show that for $\mu \in L^\infty(R^n)$ with compact support, there exists an unbounded domain Ω containing the support of μ, and a function $u \in C^{1,\alpha}_{loc}(R^n)$ satisfying

$$\begin{cases} \Delta_p u = 1 - \mu & in \quad \Omega \\ u = |\nabla u| = 0 & on \quad \partial\Omega. \end{cases}$$

1. Introduction

For a Radon measure μ with compact support, the Newtonian potential, U^μ, is defined by $U^\mu = E * \mu$ where

$$E(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & if \quad n = 2 \\ \frac{C_n}{|x|^{n-2}} & if \quad n \geq 3. \end{cases}$$

[0]*Received*: September 14 1998
1991 AMS Subject classification: Primary 31B20, 35J05, 35R35.
Key words and phrases: $p-$Laplacian, quadrature domain, existence.
Supported partially by a research grant from the University of Tehran

Here C_n is a normalizing factor so that $\Delta U^\mu = -\mu$ in R^n. In the special case, if μ is the characteristic function of a bounded domain $\Omega \subset R^n$, then the potential generated by μ is denoted by U^Ω. The bounded domain Ω is called a quadrature domain for μ (with respect to harmonic functions) if $supp(\mu) \subset \Omega$ and the following identities hold in $\Omega^c = R^n \setminus \Omega$:

$$\begin{cases} U^\mu = U^\Omega \\ \nabla U^\mu = \nabla U^\Omega. \end{cases}$$

The notion of quadrature domains has been studied by several authors among which we may refer for example to [5], [8], [9] and the references therein. There is also another equivalent formulation which may serve as the starting point for these domains. If we set $u = U^\mu - U^\Omega$, then u satisfies :

$$\begin{cases} \Delta u = \chi_\Omega - \mu & in \quad R^n \\ u = |\nabla u| = 0 & in \quad \Omega^c. \end{cases} \tag{1}$$

The converse is also easily checked to be true. Using equation (1) as the starting point, the problem is now set in the form of a free boundary problem. This form, which has also been used by several people (for example [6]), is more flexible and extends to more general operators (specially non-linear ones) and thus yields to the notion of quadrature domains with respect to these operators. In this paper we will consider this extension for the p-Laplacian, Δ_p, where $1 < p < \infty$ and where

$$\Delta_p u = div(|\nabla u|^{p-2} \nabla u).$$

To be more precise :

Definition 1.1. Suppose $1 < p < \infty$ is a fixed real number and μ is a Radon measure with compact support in R^n. The domain $\Omega \subset R^n$ (bounded or unbounded) is called a quadrature domain for μ with respect to the operator Δ_p, if it contains the support of μ and there exists a solution to the following quasi-linear elliptic equation with boundary data:

$$\begin{cases} \Delta_p u = \chi_\Omega - \mu & in \quad R^n \\ u = |\nabla u| = 0 & in \quad \Omega^c \end{cases} \tag{2}$$

The pair (u, Ω) will then be refered to as a quadrature pair for μ, and the set of all such pairs will be denoted by $QD_p(\mu)$.

Here - by imposing suitable restrictions on the measure μ - the solution is assumed to be in the space $W^{1,p}_{loc}(R^n)$, the space of Sobolev functions, and the first equation in (2) means

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \varphi)dx = \int \varphi d\mu,$$

where $\varphi \in C_0^\infty(\Omega)$ and dx denotes the Lebesgue measure in R^n.

Note that for $p = 2$, the operator Δ_p reduces to the Laplacian and thus this notion will coincide with that of the classical case. Having extended this notion for general $1 < p < \infty$, we are first faced with the problems of existence and uniqueness. In [1] a uniqueness theorem has been obtained which in our language is stated as follows: For bounded domains Ω_1 and Ω_2, if (u_1, Ω_1) and (u_2, Ω_2) are in $QD_p(\mu)$, and if $\Omega_1 \cap \Omega_2$ is convex, then $\Omega_1 = \Omega_2$. (It must be pointed out here that the situation in the classical case, $p = 2$, is no more better; see [10]). There is also another result in the general case, concerning the existence of a uniform bound for bounded quadrature domains, which intensify the idea of uniqueness of such domains in general: If μ is in the class $L^\infty(R^n)$ then there is a bounded domain $\Omega_0 \subset R^n$ such that for any pair (u, Ω) in $QD_p(\mu)$ with Ω bounded, Ω is contained in Ω_0 (see [2]).

In this paper we will consider the problem of existence for the case where $\mu \in L^\infty(R^n)$. We will prove the existence of unbounded quadrature domains for such measures while, as our construction will show, such domains are not unique for a given measure μ. It must be pointed out here that the existence of bounded quadrature domains for an arbitrary $\mu \in L^\infty$ may fail in general. This follows easily from the fact that if Ω is a bounded quadrature domain for μ and with respect to Δ_p then we must have:

$$|\Omega| = \int \mu$$

Our main tool in this direction is the calculus of variation, i.e. the minimization procedure of some kind of functionals over some subset of $W^{1,p}$. We will also make use of the well-known comparison principle which for the sake of convenience we state it here (see also [7]) :

Comparison Principle: Suppose $U \subset R^n$ is a bounded domain and let the functions $u, v \in W^{1,p}(U)$ satisfy $\Delta_p u \geq \Delta_p v$ in U. If $\limsup_{y\to x} u \leq \liminf_{y\to x} v$ for any $x \in \partial U$, then $u \leq v$ in U.

The paper is organised as follows. In the next section we will consider the problems of existence and uniqueness of minimizers for some special class of functionals. Here we will also derive some properties for these minimizers. In section 3, introducing two special barrier functions, we will show that our minimizers are bounded between these two functions and thus their limit function will exist. We will continue this section by showing that this limit function satisfy equation (2) wherein Ω is an unbounded domain.

Remark. Throughout the paper $q > 1$ denotes the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. For $R > 0$, $B_R \subset R^n$ is the ball $\{x \in R^n; |x| < R\}$. The norm of the space L^p with respect to the Lebesgue measure, dx, is denoted by $\|.\|_p$. The space $W_0^{1,p}(\Omega)$ will be the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ for a bounded domain Ω.

2. Existence and Uniqueness of Minimizers

Let B be a bounded domain. For given functions $f \in L^\infty(B)$ and $u_0 \in W^{1,p}(B)$ with $u_0 \geq 0$, consider the functional

$$J(v) = J_{B,f}(v) = \int_B \frac{1}{p}|\nabla v|^p + fv \tag{3}$$

over the set

$$\mathcal{K}(B) = \mathcal{K}_{u_0}(B) = \{v \in W^{1,p}(B), v - u_0 \in W_0^{1,p}(B), v \geq 0\}. \quad (4)$$

The following holds:

Theorem 2.1. The functional J has a unique minimizer over $\mathcal{K}(B)$.

For the convenience of the reader, we sketch below the standard proof of this theorem.

Proof. Using the inequality $\frac{1}{2^p}a^p - b^p \leq |a-b|^p$ for non-negative $a, b \in R$ together with the Poincaré and Hölder's inequalities, we have

$$J(v) \geq \frac{1}{2^p p}\int_B |\nabla v - \nabla u_0|^p - \frac{1}{p}\int_B |\nabla u_0|^p + \int_B f(v-u_0) + \int_B f u_0$$

$$\geq \alpha\|v-u_0\|_p^p - \beta\|v-u_0\|_p + \gamma$$

where $\alpha > 0$ is a constant depending only on p and the domain B, $\beta = \|f\|_\infty$ and $\gamma = \int_B (f u_0 - \frac{1}{p}|\nabla u_0|^p)$. But the real function $g(t) = \alpha t^p - \beta t + \gamma$, for positive α and non-negative β, attains its absolute minimum over $[0, +\infty)$, and thus J is bounded below over $\mathcal{K}(B)$. Now using the coersivity of J together with its lowersemicotinuity ([4], Chapter 3) and the fact that $\mathcal{K}(B)$ is a closed and convex subset of $W^{1,p}(B)$, we will obtain a minimizer for J over $\mathcal{K}(B)$. Strict convexity of J implies the uniqueness of the minimizer. □

Corollary 2.2. If $\Delta_p u_0 = f$ over B(in the sense of distributions), then u_0 itself is the minimizer of J over $\mathcal{K}(B)$.

Proof. Let u denotes the minimizer of J over $\mathcal{K}(B)$. Then for non-negative $\varphi \in C_0^\infty(B)$ we have

$$\int_B (|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + f\varphi) \geq 0. \quad (5)$$

On the other hand, u_0 also satisfies

$$\int_B (|\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\varphi + f\varphi) = 0, \tag{6}$$

for all $\varphi \in C_0^\infty(B)$. Thus for non-negative $\varphi \in C_0^\infty(B)$, combining (5) and (6), we have

$$\int_B |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \geq \int_B |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\varphi,$$

i.e. $\Delta_p u \leq \Delta_p u_0$ in B. Since $u - u_0 \in W_0^{1,p}(B)$, the comparison principle yields the inequality $u_0 \leq u$ in B. Now using the convexity of $\mathcal{K}(B)$ and the above inequality, the function $w(t) = (1-t)u + tu_0 = u + t(u_0 - u)$ will be in $\mathcal{K}(B)$ for t in some neighbourhood of zero. The minimizing property of u yields

$$0 = \frac{d}{dt}J(w(t))|_{t=0} = \int |\nabla u|^{p-2}\nabla u \cdot (\nabla u_0 - \nabla u) + f(u_0 - u). \tag{7}$$

On the other hand (6) is also true if φ is allowed to be in $W_0^{1,p}(B)$. Thus substituting φ in (6) with $u - u_0$ and subtracting from (7) we obtain

$$\int_B (|\nabla u|^{p-2}\nabla u - |\nabla u_0|^{p-2}\nabla u_0) \cdot (\nabla u - \nabla u_0) = 0.$$

The monotonicity property of the above integral together with the fact that $u - u_0 \in W_0^{1,p}(B)$ yields the result. □

In the rest of this section, we will derive some properties for minimizers of the functionals with the same form as (3). For $i = 1, 2$, let $f_i \in L^\infty(B)$ and let J_i denotes the functional J_{B,f_i} where f is replaced by f_i. Suppose also $u_i \in \mathcal{K}(B)$ is the minimizer of J_i.

Lemma 2.3. If $f_1 \leq f_2$ a.e. on B, then $u_2 \leq u_1$.

Proof. Let $v = \min(u_1, u_2)$ and $w = \max(u_1, u_2)$. Then $v, w \in \mathcal{K}(B)$ and we have:

$$\int_B \frac{1}{p}|\nabla u_1|^p + \frac{1}{p}|\nabla u_2|^p = \int_B \frac{1}{p}|\nabla v|^p + \frac{1}{p}|\nabla w|^p$$

$$\int_B f_1u_1 + f_2u_2 = \int_B f_1(u_1+u_2) + (f_2-f_1)u_2 \geq \int_B f_1(v+w) + (f_2-f_1)v$$
$$= \int_B f_1w + f_2v$$

Adding both sides to each other, we obtain:

$$J_1(u_1) + J_2(u_2) \geq J_1(w) + J_2(v)$$

and thus $J_1(u_1) = J_1(w)$ and $J_2(u_2) = J_2(v)$. Using the uniqueness of minimizers we have $u_1 = w \geq v = u_2$.$\square$

In the next lemma, we will observe the behaviour of the minimizers with respect to the domain B. For this propose, suppose $u_0 \in W^{1,p}_{loc}(R^n)$ and for $R > 0$ let the functional J_R and the set $\mathcal{K}(B_R)$ be defined respectively as (3) and (4) with the difference that B is now replaced by the ball B_R.

Lemma 2.4. Suppose u_R denotes the minimizer of J_R over $\mathcal{K}(B_R)$. If for every $R > 0$, $u_R \leq u_0$ in B_R, then u_R is decreasing as a function of R.

Proof. Let $\tilde{u}_R$ be the extension of u_R into R^n. Suppose $R_1 < R_2$. By the above extension $\tilde{u}_{R_1} \in \mathcal{K}(B_{R_2})$. Now let as before $v = \min(\tilde{u}_{R_1}, u_{R_2})$ and $w = \max(\tilde{u}_{R_1}, u_{R_2})$. By assumption $w = u_0 = \tilde{u}_{R_1}$ on $B_{R_2} \setminus B_{R_1}$. Thus we have:

$$\int_{B_{R_2}} (\frac{1}{p}|\nabla v|^p + \frac{1}{p}|\nabla w|^p + f(v+w)) = \int_{B_{R_2}} (\frac{1}{p}|\nabla v|^p + fv) +$$
$$\int_{B_{R_1}} (\frac{1}{p}|\nabla w|^p + fw) + \int_{B_{R_2}\setminus B_{R_1}} (\frac{1}{p}|\nabla u_0|^p + fu_0),$$

and also

$$\int_{B_{R_2}} (\frac{1}{p}|\nabla u_{R_2}|^p + \frac{1}{p}|\nabla \tilde{u}_{R_1}|^p + f(u_{R_2} + \tilde{u}_{R_1})) = \int_{B_{R_2}} (\frac{1}{p}|\nabla u_{R_2}|^p + fu_{R_2}) +$$
$$\int_{B_{R_1}} (\frac{1}{p}|\nabla u_{R_1}|^p + fu_{R_1}) + \int_{B_{R_2}\setminus B_{R_1}} (\frac{1}{p}|\nabla u_0|^p + fu_0).$$

Since the left hand sides of the two equations are equal, we have:

$$J_{R_2}(v) + J_{R_1}(w) = J_{R_2}(u_{R_2}) + J_{R_1}(u_{R_1})$$

which as in the previous lemma will imply that $u_{R_2} = v \leq w = u_{R_1}$. $\square$

3. Existence of Unbounded Quadrature Domains

The main theorem of this section is stated as follows:

Theorem 3.1. Suppose μ is a signed Radon measure with a support contained in a bounded domain C. If $\mu \in L^\infty(R^n)$, then $QD_p(\mu)$ contains a pair (u, Ω) with $C \subset \Omega$, Ω unbounded, $u \geq 0$ and $u \in C^{1,\alpha}_{loc}(R^n)$. Moreover $\partial\Omega$ (the free boundary) is non-empty and $\Omega = \{u > 0\}$.

In order to prove Theorem 3.1, we need two lemmas. In the first lemma we will introduce two functions which will serve as barriers for a class of minimizers and constitute an upper and a lower bound for the suitable function u in Theorem 3.1.

Lemma 3.2. For constants $M > 1$ and $r_1 > 0$, let r_2, r_3, s_1 and s_2 be defined as follows:

$$r_2 = r_1(1 + \frac{1}{M}) \quad , \quad r_3 = r_1(1 + \frac{1}{M-1}) \tag{8}$$

$$r_2 - s_1 = \frac{r_1}{2M(M+1)} \quad , \quad r_2 - s_2 = \frac{r_1}{2M} \tag{9}$$

Then the functions u_0 and $\tilde{u}_0$ defined by:

$$u_0(x) = \begin{cases} 0 & , x_1 \leq 0 \\ \frac{1}{q} {x_1}^q & , 0 < x_1 \leq r_1 \\ -\frac{(M-1)^{q-1}}{q}(r_3 - x_1)^q + \frac{r_1^q}{q}(1 + \frac{1}{M-1}) & , r_1 < x_1 \leq r_2 \\ \frac{1}{q}(x_1 - r_1)^q + \frac{r_1^q}{q}(1 + \frac{1}{M-1} - \frac{1}{(M-1)M^{q-1}}) & , r_2 < x_1 \end{cases}$$

$$\tilde{u}_0(x) = \begin{cases} 0 & , x_1 \leq s_1 \\ \frac{(1+M)^{q-1}}{q}(x_1 - s_1)^q & , s_1 < x_1 \leq r_2 \\ \frac{1}{q}(x_1 - s_2)^q - \frac{r_1^q}{q 2^q M^{q-1}(M+1)} & , r_2 < x_1 \end{cases}$$

are both in $C^1(R^n)$ and satisfy

$$\Delta_p u_0 = \chi_{\{x_1>0\}} - M\chi_S$$

$$\Delta_p \tilde{u}_0 = \chi_{\{x_1>s_1\}} + M\chi_{\tilde{S}}$$

where $S = \{r_1 < x_1 < r_2\}$ and $\tilde{S} = \{s_1 < x_1 < r_2\}$. Moreover $\tilde{u}_0 \leq u_0$ in R^n.

Proof. Note that r_1, r_2, r_3, s_1 and s_2 are in the following order:

$$r_1 < s_2 < s_1 < r_2 < r_3.$$

The fact that u_0 and $\tilde{u}_0$ are in C^1 and the value of their p-Laplacian follows easily by calculation and using the identities (8) and (9).

To prove the last inequality, note that u_0 on $\{r_1 \leq x_1 \leq r_2\}$ is concave, while $\tilde{u}_0$ on $\{s_1 \leq x_1 \leq r_2\} \subset \{r_1 \leq x_1 \leq r_2\}$ is convex. Since

$$\begin{aligned}\tilde{u}_0 \mid_{x_1=r_2} = \frac{(1+M)^{q-1}}{q}(r_2 - s_1)^q &= \frac{r_1^q}{q2^q M^q(M+1)} \\ &\leq \frac{r_1^q}{qM^q}\left(\frac{M^{q+1}-1}{M-1}\right) = u_0 \mid_{x_1=r_2},\end{aligned}$$

$\tilde{u}_0 \leq u_0$ for $x_1 \leq r_2$. On the other hand, since $r_1 < s_2$ and thus $x_1 - r_1 > x_1 - s_2$, $\lim_{x_1\to+\infty}[u_0(x) - \tilde{u}_0(x)] = +\infty$, and so for x_1 large enough $u_0 \geq \tilde{u}_0$. Since both u_0 and $\tilde{u}_0$ have the same p-Laplacian for $r_2 < x_1$, the inequality $u_0 \geq \tilde{u}_0$ holds also for $r_2 \leq x_1$.□

Note that in the structure of u_0 and $\tilde{u}_0$, all the parameters r_2, r_3, s_1 and s_2 are functions of r_1 and M. Thus choosing these two, we may construct u_0 and $\tilde{u}_0$ as in Lemma 3.2.

Now for μ and C as in Theorem 3.1, choose $M > \max(1, \|\mu\|_\infty)$ and let $r_1 > 0$ be such that

$$diam(C) \leq \frac{r_1}{2M(M+1)}.$$

Construct u_0 and $\tilde{u}_0$ as in Lemma 3.2. We may also suppose that $C \subset \tilde{S} = \{s_1 < x_1 < r_2\}$ (this is possible since by (9), $r_2 - s_1 = \frac{r_1}{2M(M+1)}$). For $R > r_2$ consider the functionals

$$J(v) = \int_{B_R} \frac{1}{p}|\nabla v|^p + (1-\mu)v,$$

$$J_0(v) = \int_{B_R} \frac{1}{p}|\nabla v|^p + (\chi_{\{x_1>0\}} - M\chi_S)v$$

over $\mathcal{K}(B_R) = \{v \in W^{1,p}(B_R); v - u_0 \in W_0^{1,p}(B_R), v \geq 0\}$, where $S = \{r_1 < x_1 < r_2\}$ as in Lemma 3.2. By Theorem 2.1, both functionals have unique minimizers over $\mathcal{K}(B_R)$. Call that of J, u_R, while according to Lemma 2.2, that of J_0 is u_0 itself.

Lemma 3.3. For any $R > r_2$, $\tilde{u}_0 \leq u_R \leq u_0$ in B_R.

Proof. Since $C \subset \tilde{S} \subset S$, the inequality, $u_R \leq u_0$ follows from Lemma 2.3. To prove the other inequality, consider u_R over $D = \{x \in B_R, s_1 < x_1\}$ and use the Euler equation for u_R, as the minimizer of J over $\mathcal{K}(B_R)$. We have

$$\Delta_p u_R \leq 1 - \mu \leq 1 + M\chi_{\tilde{S}} = \Delta_p \tilde{u}_0 \quad in\ D$$

Since $\tilde{u}_0 \leq u_R$ on ∂D, by the comparison principle, $\tilde{u}_0 \leq u_R$ on D. The fact that $\tilde{u}_0 \equiv 0$ on $B_R \setminus D$ yields the required inequality over B_R. □

We are now ready to prove Theorem 3.1:

Proof of Theorem 3.1. For $R > r_2$ suppose u_R is defined as above. By lemmas 2.4 and 3.3, u_R decreases as $R \to +\infty$. Being bounded below by the function $\tilde{u}_0$, the limit function $u = \lim_{R\to\infty} u_R$ exists and satisfies $\tilde{u}_0 \leq u \leq u_0$. If $\Omega = \{u > 0\}$, then by the inequalities $\tilde{u}_0 \leq u \leq u_0$ it is clear that Ω is unbounded, $\partial\Omega$ is non-empty and that $C \subset \tilde{S} \subset \Omega$.

Now suppose $B \subset R^n$ is a bounded domain. Choose $R_0 > 0$ large enough so that $\overline{B} \subset B_{R_0}$. For $R \geq R_0$, the minimizer u_R satisfies

$$\int_{B_R} |\nabla u_R|^{p-2}\nabla u_R \cdot \nabla\varphi + (1-\mu)\varphi \geq 0$$

for $\varphi \in C_0^\infty(B_{R_0})$ with $\varphi + u_0 \geq 0$. Since u_R is bounded above by u_0, according to [3], theorem 1, $u_R \in C^1(B_{R_0})$ and there exists a constant $c < \infty$ depending only on p, n, u_0, μ and the distance from B to the boundary of B_{R_0} such that

$$\max_{x \in B} |\nabla u_R(x)| \leq c,$$

$$|\nabla u_R(x) - \nabla u_R(y)| \leq c|x-y|^\alpha \qquad x, y \in B,$$

for some suitable $0 < \alpha < 1$, i.e. the sequence (∇u_R) is equicontinuous and uniformly bounded over B. By Arzela-Ascoli theorem, (∇u_R) contains a subsequence (denoted by (∇u_R) itself) which converges uniformly on B. Thus $u = \lim u_R \in C^1(B)$ and $\nabla u = \lim \nabla u_R$, both uniformly on B. Specially $\int_B |\nabla u_R|^{p-2} \nabla u_R \cdot \nabla \varphi \to \int_B |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi$ for any $\varphi \in C_0^\infty(B)$.

Now for $\varphi \in C_0^\infty(\Omega)$, choose B large enough so that *supp*$(\varphi) \subset B$. For $R > 0$ large enough we will have

$$\int_B |\nabla u_R|^{p-2} \nabla u_R \cdot \nabla \varphi + (1-\mu)\varphi = \int_{B_R} |\nabla u_R|^{p-2} \nabla u_R \cdot \nabla \varphi + (1-\mu)\varphi \tag{10}$$

But, since *supp*$(\varphi) \subset \Omega \subset \{u_R > 0\}$, the right hand side of (10) equals zero. Letting $R \to \infty$ and using the above argument, we will have:

$$\int_B |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + (1-\mu)\varphi = 0$$

i.e. $\Delta_p u = 1 - \mu$ in Ω.

Once more for arbitrary bounded domain $B \subset R^n$, suppose $\varphi \in C_0^\infty(B)$ is such that $u + \varphi \geq 0$. Thus we have also $u_R + \varphi \geq 0$ which, by minimizing property of u_R, will imply that

$$\int_B |\nabla u_R|^{p-2} \nabla u_R \cdot \nabla \varphi + (1-\mu)\varphi \geq 0$$

Again letting $R \to \infty$, we obtain

$$\int_B |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + (1-\mu)\varphi \geq 0$$

According to [3], theorem 1, $\tilde{u}$ has a representative in $C^{1,\alpha}_{loc}(B)$. Thus $u \in C^{1,\alpha}_{loc}(R^n)$ and $u = |\nabla u| = 0$ on $\partial\Omega$.□

Acknowledgment. The author thanks gratefully Professor Henrik Shahgholian for reading this paper and giving helpful comments.

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وجود دامنهٔ تربیع بیکران برای عملگر pـلاپلاس

در این مقاله مفهوم دامنهٔ تربیع را به عملگر pـلاپلاس Δ_p به ازای $1 < p < \infty$ تعمیم می‌دهیم. نشان خواهیم داد که به ازای هر μ اندازهٔ مطلقاً پیوسته با محمل فشرده و چگال در $L^\infty(\mathbb{R}^n)$، یک دامنهٔ بیکران Ω شامل محمل μ و یک تابع $u \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$ موجود است که در شرایط

$$\begin{cases} \Delta_p u = 1 - \mu & \text{در } \Omega, \\ u = |\nabla u| = 0 & \text{بر } \partial\Omega, \end{cases}$$

صدق کند. به این ترتیب، به ازای هر μ با شرایط فوق یک دامنهٔ تربیع بیکران وابسته به عملگر Δ_p موجود است.