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Totally Probabilistic L^p spaces

Farid Bahrami[a,*], Mehdi Mohammadbaghban[b]

[a]Department of Mathematical Sciences, Isfahan university of Technology, Isfahan 84156 83111, Iran.
[b]Department of Mathematical Sciences, Isfahan university of Technology, Isfahan 84156 83111, Iran.

Abstract

In this paper, we introduce the notion of probabilistic valued measures as a generalization of non-negative measures and construct the corresponding L^p spaces, for distributions $p > \varepsilon_0$. It is also shown that if the distribution p satisfies $p \geq \varepsilon_1$ then, as in the classical case, these spaces are complete probabilistic normed spaces.

Keywords: probabilistic normed space, probabilistic valued measure, probabilistic L^p spaces.
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1. Introduction

The idea of Probabilistic Normed spaces, briefly PN spaces, was introduced by Šerstnev in 1963 in [7] and [8], who replaced $\mathbb{R}^+$, the set of all non-negative real numbers, with the elements of Δ^+, certain subset of extended distribution functions, as the target space of the norm function. These spaces and their related notions were then studied by many authors among which we may refer to [1], [3], [5], [9], [10] and the text [6]. There is also a generalization of this notion introduced in [2]. However, in this paper we consider probabilistic normed spaces still in the sence of Šerstnev. Using his idea, we introduce here the notion of probabilistic valued measures and corresponding L^p spaces for a distribution function p. The title of the paper arises from the fact, that, not only the measure and the exponent p have probabilistic natures, but also the elements of the spaces L^p, introduced in the last section, are functions with values in a certain probabilistic normed space.

We first recall some definitions and notations. Let Δ be the set of all extended distribution functions, i.e. the set of all non-decreasing and left-continuous functions $F : \mathbb{R} \to [0, 1]$, and let $D \subset \Delta$ be the set of all $F \in \Delta$ with $\inf F(\mathbb{R}) = 0$ and $\sup F(\mathbb{R}) = 1$.
The set $[-\infty, +\infty]$ can be embedded in Δ by the map $r \mapsto \varepsilon_r$, where $\varepsilon_r = \chi_{(r,+\infty)}$ for $r \in \mathbb{R}$ and

*Corresponding author
This work is partially supported by a grant from Isfahan University of Technology
Email addresses: fbahrami@cc.iut.ac.ir (Farid Bahrami), baghban@math.iut.ac.ir (Mehdi Mohammadbaghban)

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$\varepsilon_{-\infty} = \chi_{\mathbb{R}}$ and $\varepsilon_{+\infty} = \chi_{\emptyset}$. Here χ_A denotes the characteristic function of a set A.
There is a metric on Δ, known as Levy metric, which for every $F, G \in \Delta$ is defined as follows

$$d_L(F, G) = \inf\{h > 0 | F(t-h) - h \leq G(t) \leq F(t+h) + h \text{ for all } t \in \mathbb{R}\}$$

One can define a partial ordering on Δ by reverse point-wise ordering of the real valued functions. Hence, for $F, G \in \Delta$, we say $F \leq G$ if $F(t) \geq G(t)$ for all $t \in \mathbb{R}$. According to this order, the non-negative elements of Δ, denoted by Δ^+, is defined equal to the set $\{F \in \Delta | \varepsilon_0 \leq F\}$. We also let $D^+ \subset D$ be defined equal to $\Delta^+ \cap D$.
A triangle function is a map $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, non-decreasing in each variable and with ε_0 as identity. The most famous triangle functions are those defined by t-norms. A t-norm is a binary operation $T : [0,1] \times [0,1] \to [0,1]$ that is associative, commutative, non-decreasing in each variable and with 1 as identity. The most important t-norm is Min defined for $a, b \in [0,1]$ by $\text{Min}(a,b) = \min\{a,b\}$. The importance of this t-norm will become clear in the next section. Corresponding to a t-norm T, the map $\tau_T : \Delta^+ \times \Delta^+ \to \Delta^+$ defined for every $F, G \in \Delta^+$ and $t \in \mathbb{R}$ by

$$\tau_T(F, G)(t) := \sup\{T(F(r), G(s)) | r + s = t\}$$

is a triangle function. In the definition of a PN space, we need also the multiplications of elements of Δ^+ by non-negative real numbers. More generally, for $F \in \Delta$ and $r \in \mathbb{R}^+$, we denote by $r \cdot F$ the distribution function defined by $r \cdot F(t) = F(t/r)$, if $r \neq 0$. If $r = 0$ then $r \cdot F$ is defined equal to ε_0. We are now ready to give the definition of a PN space that was introduced by Šerstnev in 1963.

Definition 1.1. *Let X be a real vector space and τ be a triangle function. A map $\|\cdot\| : X \to \Delta^+$ is called a probabilistic norm on X if it satisfy the following properties*

1. $\|x\| = \varepsilon_0$ *if and only if* $x = 0$,
2. $\|rx\| = |r| \cdot \|x\|$ *for* $r \in \mathbb{R}$ *and* $x \in X$,
3. $\|x + y\| \leq \tau(\|x\|, \|y\|)$ *for* $x, y \in X$.

In this case $(X, \tau, \|\cdot\|)$ is called a probabilistic normed space or briefly a PN space.

Let $(X, \tau, \|\cdot\|)$ be a PN space. For $x \in X$ and $r \geq 0$, if $B(x,r)$ is defined as follows

$$B(x, r) := \{y \in X | d_L(\|x - y\|, \varepsilon_0) < r\}$$

then the family $\{B(x,r) | x \in X, r > 0\}$ forms a basis for a topology on X which is called the strong topology corresponding to the probabilistic norm $\|\cdot\|$.

In the next section, we first recall one of the important properties of the t-norm Min. Then substitute Δ with a new class of functions, which will appear to be more flexible, and then translate the operations and order on Δ in this new class. We will also introduce a certain PN space which plays a central roll in defining the probabilistic L^p spaces. In the last section, we introduce the concept of probabilistic valued measures and integrals which will lead us to probabilistic L^p spaces, with $p \in D^+$, a distribution function between ε_1 and ε_∞.

2. Some Preliminaries

Let τ_T be the triangle function corresponding to a t-norm T. One of the most important relations which is needed in this paper is

$$(\alpha + \beta) \cdot F = \tau_T(\alpha \cdot F, \beta \cdot F), \tag{2.1}$$

for $\alpha, \beta \in \mathbb{R}^+$ and $F \in \Delta^+$. It is easily seen that this equality holds if $T = \mathrm{Min}$. The converse has also been proved to be true in [4], i.e.

Theorem 2.1. *Let $T : [0,1] \times [0,1] \to [0,1]$ be a continuous t-norm. Then the relation (2.1) is satisfied for all $F \in \Delta^+$ and $\alpha, \beta \in \mathbb{R}^+$ if and only if T equals the t-norm* Min.

In the remaining of this paper, we use $F \oplus G$ instead of $\tau_{\mathrm{Min}}(F,G)$, for $F, G \in \Delta^+$. Hence, for distribution functions $F, G \in \Delta^+$, the distribution function $F \oplus G : (-\infty, +\infty) \to [0,1]$ is defined by

$$F \oplus G(t) = \sup\{\min(F(r), G(s)) \mid r + s = t\}, \quad \forall t \in \mathbb{R}$$

Let V be the set of all functions $f : (0,1) \to [-\infty, +\infty]$. For $F \in \Delta$, let $\widehat{F} \in V$ be defined, for all $a \in (0,1)$, by

$$\widehat{F}(a) := \begin{cases} \sup F^{-1}((0,a)) & F^{-1}((0,a)) \neq \emptyset, \\ -\infty & F^{-1}((0,a)) = \emptyset. \end{cases}$$

It is easily seen that, for each $F \in \Delta$, $\widehat{F}$ is a non-decreasing left continuous function on $(0,1)$. Also the following relations are true for all $t \in \mathbb{R}$ and $a \in (0,1)$.

$$\widehat{F}(a) < t \;\Rightarrow\; a \leq F(t) \tag{2.2}$$

$$t < \widehat{F}(a) \;\Rightarrow\; F(t) < a \tag{2.3}$$

$$F(t) < a \;\Rightarrow\; t \leq \widehat{F}(a) \tag{2.4}$$

Moreover, $F(\widehat{F}(a)) \leq a$ and $\widehat{F}(F(t)) \leq t$. From these relations one may characterize the finite values of $\widehat{F}$ as follows. For $a \in (0,1)$ and $t_0 \in \mathbb{R}$, $\widehat{F}(a) = t_0$ if and only if

$$\forall \epsilon > 0, \quad F(t_0 - \epsilon) < a \quad \text{and} \quad F(t_0 + \epsilon) \geq a$$

For $f \in V$, suppose $\widetilde{f} : (-\infty, +\infty) \to [0,1]$ is defined as follows.

$$\widetilde{f}(t) = \begin{cases} \sup f^{-1}((-\infty,t)) & f^{-1}((-\infty,t)) \neq \emptyset, \\ 0 & f^{-1}((-\infty,t)) = \emptyset. \end{cases}$$

It is easily verified that $\widetilde{f} \in \Delta$, for all $f \in V$, and that if $f \in V$ is non-decreasing and left continuous then

$$\forall \epsilon > 0, \quad f(\widetilde{f}(t) - \epsilon) < t \quad \text{and} \quad f(\widetilde{f}(t) + \epsilon) \geq t$$

for all $t \in \mathbb{R}$.

Lemma 2.2. *The map $\widehat{\ }: \Delta \to V$, given by $F \mapsto \widehat{F}$, is one to one. Moreover, $(\widehat{F}\,\widetilde{)} = F$, for all $F \in \Delta$.*

Proof . For $F \in \Delta$, it is shown that $(\widehat{F}\,\widetilde{)} = F$ which proves also the first assertion. We denote $\widehat{F}$ by f, and show that $\widetilde{f} = F$. Let $t_0 \in \mathbb{R}$. As it was mentioned above,

$$\forall \epsilon > 0, \quad f(\widetilde{f}(t_0) - \epsilon) < t_0$$

or $\widehat{F}(\widetilde{f}(t_0) - \epsilon) < t_0$. Hence, by (2.2),

$$\widetilde{f}(t_0) - \epsilon \leq F(t_0)$$

Since this is true for all $\epsilon > 0$, we have $\widetilde{f}(t_0) \leq F(t_0)$. Conversely, for each $\epsilon > 0$,

$$t_0 - \epsilon < t_0 \leq f(\widetilde{f}(t_0) + \epsilon) = \widehat{F}(\widetilde{f}(t_0) + \epsilon)$$

Therefore, by (2.3), $F(t_0 - \epsilon) < \widetilde{f}(t_0) + \epsilon$. Using the left continuity of F, we have

$$F(t_0) = \lim_{\epsilon \to 0^+} F(t_0 - \epsilon) \leq \lim_{\epsilon \to 0^+} \left(\widetilde{f}(t_0) + \epsilon\right) = \widetilde{f}(t_0)$$

which completes the proof. □

Let $\widehat{\Delta} := \{\widehat{F} \mid F \in \Delta\}$. As we have seen the map$\widehat{\ }: \Delta \to \widehat{\Delta} \subset V$ is a one to one and on to map with the inverse$\widetilde{\ }: \widehat{\Delta} \to \Delta$. As the following lemma proves, $\widehat{\Delta}$ comprises all elements of V which are non-decreasing and left continuous.

Lemma 2.3. *Let $f \in V$ be a non-decreasing and left continuous function. Then there exists (a unique) $F \in \Delta$ for which $\widehat{F} = f$.*

Proof . Let $F := \widetilde{f}$. Then $F \in \Delta$. We show that $\widehat{F} = f$. For $a_0 \in (0, 1)$ and each $t \in \mathbb{R}$ with $t < \widehat{F}(a_0)$, by (2.3), we have $F(t) < a_0$ or $\widetilde{f}(t) < a_0$. Thus $t \leq f(a_0)$. Since this is true for all $t < \widehat{F}(a_0)$, we obtain the inequality $\widehat{F}(a_0) \leq f(a_0)$. If $\widehat{F}(a_0) < f(a_0)$ then for $t_0 \in \mathbb{R}$ with $\widehat{F}(a_0) < t_0 < f(a_0)$, by left continuity of f, there exists $b < a_0$ such that $t_0 < f(b)$. Hence

$$F(t_0) = \widetilde{f}(t_0) \leq b < a_0$$

which, by (2.4), implies that $t_0 \leq \widehat{F}(t_0)$ which contradicts the choice of t_0. Thus $\widehat{F}(a_0) = f(a_0)$, for all $a \in (0, 1)$. □

By Lemmas 2.2 and 2.3, the map $\widehat{\ }$ defines a one to one correspondence between the set Δ and the set of all non-decreasing left continuous functions $f : (0, 1) \to [-\infty, +\infty]$, i.e. $\widehat{\Delta}$. In the following two lemmas, we consider the order and operations on the set $\widehat{\Delta}$ which correspond to those on Δ.

Lemma 2.4. *For $F, G \in \Delta$, $F \leq G$ if and only if $\widehat{F}(a) \leq \widehat{G}(a)$, for all $a \in (0, 1)$, i.e. the corresponding order on $\widehat{\Delta}$ is simply the natural order on the (extended) real valued functions.*

Proof . Let $F, G \in \Delta$ and suppose $F \leq G$. Hence, according to the order defined on the set Δ, $F(t) \geq G(t)$ for all $t \in \mathbb{R}$. Therefore, for each $a \in (0, 1)$,

$$\begin{aligned} \widehat{F}(a) &= \sup\{t \in \mathbb{R} \mid F(t) < a\} \\ &\leq \sup\{t \in \mathbb{R} \mid G(t) < a\} = \widehat{G}(a) \end{aligned}$$

Conversely, suppose $\widehat{F} \leq \widehat{G}$ and $F \not\leq G$. Then there exists $t \in \mathbb{R}$ such that $F(t) < G(t)$. By left continuity of G there exists $s < t$ such that $F(t) < G(s) \leq G(t)$. Hence for $a \in (0, 1)$ with $F(t) < a < G(s)$, we have $\widehat{G}(a) \leq s < t \leq \widehat{F}(a)$ which contradicts the assumption. □

Lemma 2.5. *Let $F, G \in \Delta$ and $\lambda \in [0, \infty)$. Then $(\widehat{F \oplus G}) = \widehat{F} + \widehat{G}$ and $(\widehat{\lambda \cdot F}) = \lambda \widehat{F}$.*

Proof . For $F, G \in \Delta$ and $a \in (0, 1)$, we denote $\widehat{F}(a)$ and $\widehat{G}(a)$, respectively, by F_a and G_a. By (2.2) and (2.3), we have $F(F_a + \varepsilon), G(G_a + \varepsilon) \in [a, 1]$ and $F(F_a - \varepsilon), G(G_a - \varepsilon) \in [0, a)$, for each $\varepsilon > 0$, and

$$\begin{aligned} F \oplus G \quad & (F_a + G_a - 2\varepsilon) \\ &= \sup\{\min(F(F_a - \varepsilon - u), G(G_a - \varepsilon + u)) \mid u \in \mathbb{R}\} \\ &\leq \max(F(F_a - \varepsilon), G(G_a - \varepsilon)) < a \end{aligned}$$

And

$$
\begin{aligned}
F \oplus G \qquad & (F_a + G_a + 2\varepsilon) \\
&= \sup\{\min(F(F_a + \varepsilon - u), G(G_a + \varepsilon + u)) \mid u \in \mathbb{R}\} \\
&\geq \min(F(F_a + \varepsilon), G(G_a + \varepsilon)) \geq a
\end{aligned}
$$

Hence $(\widehat{F \oplus G})(a) = F_a + G_a = \widehat{F}(a) + \widehat{G}(a)$.

For $\lambda \in (0, \infty)$,

$$
\begin{aligned}
(\widehat{\lambda \cdot F})(a) &= \{t \in [-\infty, +\infty] \mid (\lambda \cdot F)(t) < a\} \\
&= \{t \in [-\infty, +\infty] \mid F(t/\lambda) < a\} \\
&= \{\lambda t \in [-\infty, +\infty] \mid F(t) < a\} = \lambda \widehat{F}(a)
\end{aligned}
$$

Therefore, $(\widehat{\lambda \cdot F}) = \lambda \widehat{F}$. $\square$

Let $(X, \tau_{\mathrm{Min}}, \|\cdot\|)$ be a PN space. If the composite map $X \xrightarrow{\|\cdot\|} \Delta^+ \xrightarrow{\widehat{\ }} \widehat{\Delta}$ is still denoted by the same notation $\|\cdot\|$ then, by the previous two lemmas, the map $\|\cdot\| : X \to \widehat{\Delta}$ satisfies the following relations.

1. $\|x\| = 0$ if and only if $x = 0$,
2. $\|rx\| = |r|\,\|x\|$ for $r \in \mathbb{R}$ and $x \in X$,
3. $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in X$,

where all the operations are understood to be point-wise. The converse is also true, i.e. if a map $\|\cdot\| : X \to \widehat{\Delta}$ satisfies the above three conditions and, for each $x \in X$ the map $(\widetilde{\|x\|}) : \mathbb{R} \to [0, 1]$ is again denoted by $\|x\|$, then $(X, \tau_{\mathrm{Min}}, \|\cdot\|)$ is a PN space.

As it will be seen, the set $\widehat{\Delta}$ provides us with more facilities. Hence our aim will be to replace the set Δ with this new set of functions. According to these consideration, from now on, whenever we talk about a PN space we mean a vector space X and a map $\|\cdot\| : X \to \widehat{\Delta^+}$ which satisfies conditions (1)-(3) above. As it is known, the strong topology of a PN space is metrizable. In order to obtain the structure of this metric in this new language, we need the following two lemmas.

Lemma 2.6. *For $F \in \Delta^+$,*

$$
d_L(F, \varepsilon_0) = \begin{cases} \inf\{h \in (0,1) \mid \widehat{F}(1-h) < h\} & \text{if this set is non-empty,} \\ 1 & \text{otherwise.} \end{cases}
$$

Proof . According to the definition of the Levy metric, for each $F \in \Delta^+$ we have

$$
\begin{aligned}
d_L(F, \varepsilon_0) &= \inf\{h > 0 \mid F(t-h) - h \leq \varepsilon_0(t) \leq F(t+h) + h, \quad \forall t \in \mathbb{R}\} \\
&= \inf\{h > 0 \mid \varepsilon_0(t) \leq F(t+h) + h, \quad \forall t > 0\} \\
&= \inf\{h > 0 \mid 1 \leq F(t+h) + h, \quad \forall t > 0\} \\
&= \inf\{h > 0 \mid 1 - h \leq F(t), \quad \forall t > h\} \leq 1
\end{aligned}
$$

If $d_L(F, \varepsilon_0) = 1$ then for each $t < 1$, $F(t) = 0$ from which it follows that $\widehat{F}(a) \geq 1$, for each $a \in (0, 1)$. Therefore, $\{h \in (0, 1) \mid \widehat{F}(1-h) < h\} = \emptyset$.

Now suppose this set is non-empty and therefore $d_L(F, \varepsilon_0) < 1$. In this case, for $r \in (0, 1)$ with $r > \inf\{h \in (0, 1) \mid \widehat{F}(1-h) < h\}$ there exists $h_0 < r$ such that $\widehat{F}(1-h_0) < h_0$. Hence by (2.2),

$$
1 - r < 1 - h_0 \leq F(h_0) \leq F(r) \leq F(t)
$$

for all $t > r$. Therefore, $d_L(F, \varepsilon_0) \le r$. By arbitrariness of $r > 0$, we have $d_L(F, \varepsilon_0) \le \inf\{h > 0 \mid \widehat{F}(1-h) < h\}$.

Conversely, if $r > 0$ satisfies $r > d_L(F, \varepsilon_0)$ then there exists $h_0 < r$ with $1 - h_0 \le F(t)$, for all $t > h_0$. By (2.3),

$$\widehat{F}(1-h_0) \le t$$

Since this is true for all $t > h_0$, we have $\widehat{F}(1-h_0) \le h_0$. Without loss of generality, we may assume that $\widehat{F}(1-h_0) < h_0$. Thus

$$\inf\{h > 0 \mid \widehat{F}(1-h) < h\} \le h_0 < r$$

from which it follows that $\inf\{h > 0 \mid \widehat{F}(1-h) < h\} \le d_L(F, \varepsilon_0)$. □

Lemma 2.7. *Let $F, G \in \Delta^+$.*

(i) *If $F \le G$ then $d_L(F, \varepsilon_0) \le d_L(G, \varepsilon_0)$.*
(ii) $d_L(F \oplus G, \varepsilon_0) \le d_L(F, \varepsilon_0) + d_L(G, \varepsilon_0)$.

Proof . Part (i) follows easily from the definition of the order on the set Δ and the previous lemma.

To prove part (ii), let $F, G \in \Delta^+$. If $d_L(F, \varepsilon_0) + d_L(G, \varepsilon_0) \ge 1$ then clearly $d_L(F \oplus G, \varepsilon_0) \le d_L(F, \varepsilon_0) + d_L(G, \varepsilon_0)$. So suppose $d_L(F, \varepsilon_0) + d_L(G, \varepsilon_0) < 1$. For $\epsilon > 0$ with $\epsilon < 1 - (d_L(F, \varepsilon_0) + d_L(G, \varepsilon_0))$, there are $h_1 < d_L(F, \varepsilon_0) + \frac{\epsilon}{2}$ and $h_2 < d_L(G, \varepsilon_0) + \frac{\epsilon}{2}$ such that

$$\widehat{F}(1-h_1) < h_1 \quad \text{and} \quad \widehat{G}(1-h_2) < h_2$$

Therefore,

$$\widehat{F}(1-(h_1+h_2)) + \widehat{G}(1-(h_1+h_2)) \le \widehat{F}(1-h_1) + \widehat{G}(1-h_2) < h_1 + h_2$$

from which, using Lemmas 2.5 and 2.6, it follows that

$$\begin{aligned} d_L(F \oplus G, \varepsilon_0) &= \inf\{h \in (0,1) \mid \widehat{F \oplus G}(1-h) < h\} \\ &= \inf\{h \in (0,1) \mid (\widehat{F} + \widehat{G})(1-h) < h\} \\ &\le h_1 + h_2 < d_L(F, \varepsilon_0) + d_L(G, \varepsilon_0) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, the result is obtained. □

As a direct application of the previous lemma, we obtain the structure of the metric which induces the strong topology of a PN space.

Corollary 2.8. *Let X be a real vector space and $\|\cdot\| : X \to \widehat{\Delta^+}$ be a probabilistic norm on it. If $d : X \times X \to \mathbb{R}^+$ is defined, for all $x, y \in X$, by*

$$d(x,y) = \begin{cases} \inf\{h \in (0,1) \mid \|x-y\|(1-h) < h\} & \text{if this set is non-empty,} \\ 1 & \text{otherwise.} \end{cases}$$

then d is a metric on X which induces the strong topology of the PN space $(X, \|\cdot\|)$.

We recall that V is the set of all functions $f:(0,1)\to[-\infty,+\infty]$. Clearly V itself is not a vector space. However, it contains a vector space which will play a crucial roll in this paper. For $f\in V$, let $|f|_{\sup}:(0,1)\to[0,+\infty]$ be defined by

$$\forall a\in(0,1),\quad |f|_{\sup}(a)=\sup\{|f(c)|\mid c\in(0,a)\}$$

It is easily seen that $|f|_{\sup}$ is a non-decreasing and left continuous function on $(0,1)$, i.e. $|f|_{\sup}\in\widehat{\Delta^+}$, for each $f\in V$. Let $V_{\sup}\subset V$ be defined as follows.

$$V_{\sup}:=\{f\in V\mid |f|_{\sup}\in\widehat{D^+}\}$$

Note that $D^+\subset\Delta^+$ is the set of all distribution functions $F:\mathbb{R}\to[0,1]$ with $F(0)=0$ and $\lim_{t\to+\infty}F(t)=1$. Therefore, $\widehat{D^+}\subset\widehat{\Delta}$ comprises all non-decreasing left continuous finite valued functions $f:(0,1)\to[0,+\infty)$. Therefore, each element of $V_{\sup}$ is a finite valued function on $(0,1)$.

Proposition 2.9. *The pair $(V_{\sup},|\cdot|_{\sup})$ is a complete PN space, i.e. every Cauchy sequence in this PN space converges.*

Proof . The fact that $|\cdot|_{\sup}:V_{\sup}\to\widehat{D^+}\subset\widehat{\Delta^+}$ is a probabilistic norm on $V_{\sup}$ is easily proved. According to Corollary 2.8, the function $d_{\sup}:V_{\sup}\times V_{\sup}\to[0,+\infty)$ given, for all $f,g\in V_{\sup}$, by

$$d_{\sup}(f,g)=\inf\{h\in(0,1)\mid |f-g|_{\sup}(1-h)<h\} \tag{2.5}$$

if this set in non-empty, and $d_{\sup}(f,g)=1$ otherwise, is a metric which induces the strong topology of the PN space $(V_{\sup},|\cdot|_{\sup})$.

Suppose $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in this PN space. Hence for each $\epsilon>0$ there exists $N\in\mathbb{N}$ such that

$$d_{\sup}(f_n,f_m)=\inf\{h\in(0,1)\mid |f_n-f_m|_{\sup}(1-h)<h\}<\epsilon$$

for all $m,n\geq N$. Now for $a\in(0,1)$, we choose $\epsilon>0$ with $\epsilon<1-a$. Then there exists $N\in\mathbb{N}$ and $h<\epsilon$ such that

$$|f_n-f_m|_{\sup}(1-h)<h$$

for all $m,n\geq N$. Since $h<\epsilon<1-a$ we have $a<1-h$. Hence, for all $m,n\geq N$ and all $c\in(0,a)$,

$$\begin{aligned}|f_n(c)-f_m(c)| &\leq \sup\{|f_n(c)-f_m(c)|\mid c\in(0,a)\}\\ &= |f_n-f_m|_{\sup}(a)\leq|f_n-f_m|_{\sup}(1-h)<h<\epsilon\end{aligned}$$

i.e. the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on $(0,a)$, for each $a\in(0,1)$. Thus there is a function $f:(0,1)\to\mathbb{R}$ such that $f_n\to f$, uniformly on the interval $(0,a)$, for each $a\in(0,1)$. Using this uniform convergence, it is now easily seen that $f\in V_{\sup}$ and that $d_{\sup}(f_n,f)\to0$. □

Remark 2.10. The definition of the metric $d_{\sup}$, introduced in the proof of the previous proposition, can be extended to $\widehat{\Delta^+}$. Hence for $f,g\in\widehat{\Delta^+}$ we define $d_{\sup}(f,g)$ as in (2.5), with the convention that $\infty-\infty:=0$. As in the above proof, it is seen that a sequence $(f_n)_{n\in\mathbb{N}}$ in $\widehat{\Delta^+}$ converges, under this metric, to some $f\in\widehat{\Delta^+}$ if and only if this sequence converges uniformly on the interval $(0,a)$ to f, for each $a\in(0,1)$.

3. Probabilistic L^p spaces

In this section, we first define the notion of a probabilistic valued measure. It should be noted that in [4], the authors have defined a similar notion which extends the finite valued measures. As it is seen, the concept of probabilistic valued measure, defined in this paper, is an extention of a non-negative measure (not necessarily finite) to its probabilistic case. Moreover, the concept of probabilistic L^p spaces introduced here differs, in principle, with those in [4]. The differences arise, firstly because of the different topologies on the target spaces of probabilistic valued measures here and in [4], and secondly, as we will see, the exponent p in this paper is chosen itself to have a probabilistic nature.

Let (Ω, Σ) be a measurable space, i.e. Ω be a non-empty set and Σ be a σ-algebra of its subsets.

Definition 3.1. *A set valued function $\mu : \Sigma \to \widehat{\Delta^+}$ is called a probabilistic valued measure, if $\mu(\emptyset) = 0$ and*

$$\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

for all disjoint countable family $\{A_n \mid n \in \mathbb{N}\} \in \Sigma$. Here the convergence of the series is understood to be in the metric space $(\widehat{\Delta^+}, d_{\sup})$.

Let μ be a probabilistic valued measure on the measurable space (Ω, Σ). If for each $a \in (0, 1)$, the set function $\mu_a : \Sigma \to [0, +\infty]$ is defined by $\mu_a(A) := \mu(A)(a)$, then, according to Remark 2.10, μ_a is a non-negative measure on (Ω, Σ). Moreover, for all $a, b \in (0, 1)$ with $a < b$, $\mu_a(A) \leq \mu_b(A)$, for all $A \in \Sigma$.

For a function $\phi : \Omega \to V$ and $a \in (0, 1)$, let $\phi_a : \Omega \to [-\infty, +\infty]$ be defined for all $\omega \in \Omega$, by

$$\phi_a(\omega) := \phi(\omega)(a)$$

Suppose M_Ω denotes the set of all measurable functions on Ω with values in $[-\infty, +\infty]$ and let $M_{\Omega,V}$ be defined as follows.

$$M_{\Omega,V} := \{\phi : \Omega \to V \mid \forall a \in (0, 1), \quad \phi_a \in M_\Omega\}$$

For a function $\phi : \Omega \to V$, we let $|\phi| : \Omega \to V$ be defined as

$$\forall \omega \in \Omega, \quad |\phi|(\omega) := |\phi(\omega)|,$$

where $|\cdot| : V \to V$ is defined for each $f : (0, 1) \to [-\infty, +\infty]$ in the natural manner. Finally, we call $\phi : \Omega \to V$ non-negative if $\phi(\omega) : (0, 1) \to [-\infty, +\infty]$ is a non-negative function, for each $\omega \in \Omega$.

Definition 3.2. *Let μ be a probabilistic valued measure on the measurable space (Ω, Σ). For a non-negative function $\phi \in M_{\Omega,V}$, the integral of ϕ with respect to μ on Ω, denoted by $\int_\Omega \phi \, d\mu$, is defined as an element of V as follows.*

$$\forall a \in (0, 1), \quad (\int_\Omega \phi \, d\mu)(a) := \int_\Omega \phi_a \, d\mu_a$$

A function $\phi \in M_{\Omega,V}$ is called integrable if $\int_\Omega |\phi| \, d\mu \in V_{\sup}$.

For a function $\phi \in M_{\Omega,V}$, it is clear that $|\phi|$ is a non-negative element of $M_{\Omega,V}$. Moreover, for each $p \in \widehat{D^+}$, the function $|\phi|^p : \Omega \to V$ defined for each $\omega \in \Omega$ and $a \in (0,1)$ as

$$|\phi|^p(\omega)(a) := |\phi(\omega)(a)|^{p(a)} = |\phi_a(\omega)|^{p(a)}$$

belongs also to $M_{\Omega,V}$.

Definition 3.3. *Let μ be a probabilistic valued measure on (Ω, Σ). For $p \in \widehat{D^+}$, the set $L^p(\Omega, \mu)$ is defined as follows.*

$$L^p(\Omega, \mu) := \{\phi \in M_{\Omega,V} \mid \int_\Omega |\phi|^p \, d\mu \in V_{\sup}\}$$

Since $V_{\sup}$ consists of finite valued functions on $(0,1)$, a function $\phi \in M_{\Omega,V}$ belongs to $L^p(\Omega, \mu)$ if and only if for each $a \in (0,1)$,

$$(\int_\Omega |\phi|^p \, d\mu)(a) = \int_\Omega |\phi_a|^{p(a)} \, d\mu_a < +\infty$$

i.e. if and only if $\phi_a \in L^{p(a)}(\Omega, \mu_a)$, for each $a \in (0,1)$. Hence each $\phi_a : \Omega \to [-\infty, +\infty]$ is μ_a-a.e. finite on Ω. Therefore, for ϕ and ψ in $L^p(\Omega, \mu)$, $\phi + \psi$ is definable as a function from Ω to V and, using the linear structure of $L^{p(a)}(\Omega, \mu_a)$, belongs to $L^p(\Omega, \mu)$. It follows that $L^p(\Omega, \mu)$ is a linear space. As in the classical theory, it is necessary to indentify all ϕ in this space which are almost everywhere identical. Hence we define a relation $\sim$ on $L^p(\Omega, \mu)$ as follows. For $\phi, \psi \in L^p(\Omega, \mu)$, we say $\phi \sim \psi$ if $\int_\Omega |\phi - \psi|^p \, d\mu = 0$, the constant function 0 on $(0,1)$. It is easily seen that $\sim$ is an equivalence relation. We denote the set of all equivalent classes still by $L^p(\Omega, \mu)$.

Theorem 3.4. *Let $p \in \widehat{D^+}$ satisfies $p \geq 1$. If $\| \cdot \|_p : L^p(\Omega, \mu) \to \widehat{D^+} \subset \widehat{\Delta^+}$ is defined by*

$$\forall \phi \in L^p(\Omega, \mu), \quad \|\phi\|_p := \Big|(\int_\Omega |\phi|^p d\mu)^{\frac{1}{p}}\Big|_{\sup}$$

then $(L^p(\Omega, \mu), \mu)$ is a PN space.

Proof . For $\phi \in L^p(\Omega, \mu)$ suppose $\|\phi\|_p = 0$. Then, as an element of $V_{\sup}$, $(\int_\Omega |\phi|^p \, d\mu)^{\frac{1}{p}} = 0$. Hence $\int_\Omega |\phi|^p \, d\mu = 0$. Therefore, according to the equivalence relation defined above, $\phi = 0 \in L^p(\Omega, \mu)$.

For $\phi, \psi \in L^p(\Omega, \mu)$ and for each $a \in (0,1)$, by the triangle inequality of the norm in the normed space $L^{p(a)}(\Omega, \mu_a)$,

$$(\int_\Omega |\phi_a + \psi_a|^{p(a)} d\mu_a)^{\frac{1}{p(a)}} \leq (\int_\Omega |\phi_a|^{p(a)} d\mu_a)^{\frac{1}{p(a)}} + (\int_\Omega |\psi_a|^{p(a)} d\mu_a)^{\frac{1}{p(a)}}$$

Hence, as elements of $V_{\sup}$,

$$(\int_\Omega |\phi + \psi|^p d\mu)^{\frac{1}{p}} \leq (\int_\Omega |\phi|^p d\mu)^{\frac{1}{p}} + (\int_\Omega |\psi|^p d\mu)^{\frac{1}{p}}$$

therefore,

$$\begin{aligned} \|\phi + \psi\|_p = \Big|(\int_\Omega |\phi + \psi|^p d\mu)^{\frac{1}{p}}\Big|_{\sup} &\leq \Big|(\int_\Omega |\phi|^p d\mu)^{\frac{1}{p}} + (\int_\Omega |\psi|^p d\mu)^{\frac{1}{p}}\Big|_{\sup} \\ &\leq \Big|(\int_\Omega |\phi|^p d\mu)^{\frac{1}{p}}\Big|_{\sup} + \Big|(\int_\Omega |\psi|^p d\mu)^{\frac{1}{p}}\Big|_{\sup} \\ &= \|\phi\|_p + \|\psi\|_p \end{aligned}$$

the relation $\|\lambda\phi\|_p = |\lambda| \, \|\phi\|_p$, for $\lambda \in \mathbb{R}$ and $\phi \in L^p(\Omega, \mu)$, is obtained easily. □

As our last theorem, we prove the completeness of this PN space.

Theorem 3.5. *For $p \in \widehat{D^+}$ with $p \geq 1$, the PN space $(L^p(\Omega\mu), \|\cdot\|_p)$ is complete.*

Proof . Let $(\phi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(L^p(\Omega, \mu), \|\cdot\|_p)$. Hence for each $\epsilon > 0$

$$\exists N \in \mathbb{N},\ \forall m, n \geq N, \quad d(\phi_n, \phi_m) < \epsilon, \tag{3.1}$$

where

$$d(\phi_n, \phi_m) = \inf\{h \in (0,1) \mid \|\phi_n - \phi_m\|_p(1-h) < h\}.$$

For each $a \in (0,1)$, we choose $b \in (0,1)$ with $a < b$. For $0 < \epsilon < 1-b$, by (3.1), there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(\phi_n, \phi_m) < \epsilon$. Hence there is $h < \epsilon$ with $\|\phi_n - \phi_m\|_p(1-h) < h$. Using the relations $0 < h < \epsilon < 1-b$, we have $b < 1-h$, from which it follows that

$$\|\phi_n - \phi_m\|_p(b) \leq \|\phi_n - \phi_m\|_p(1-h) < h < \epsilon$$

and consequently,

$$\sup_{c\in(0,b)} \Big(\int_\Omega |\phi_n - \phi_m|^p \, d\mu\Big)^{\frac{1}{p}}(c) < \epsilon.$$

Since $a < b$, we obtain

$$\Big(\int_\Omega |\phi_n - \phi_m|^p \, d\mu\Big)^{\frac{1}{p}}(a) = \Big(\int_\Omega |\phi_{n,a} - \phi_{m,a}|^{p(a)} \, d\mu\Big)^{\frac{1}{p(a)}} < \epsilon,$$

for all $m, n \geq N$. Therefore, for each $a \in (0,1)$, the sequence $(\phi_{n,a})_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $L^{p(a)}(\Omega, \mu_a)$, hence convergent, i.e. for each $a \in (0,1)$ there exists a function $\phi_{(a)} \in L^{p(a)}(\Omega, \mu_a)$ such that $\|\phi_{n,a} - \phi_{(a)}\|_{L^{p(a)}(\Omega,\mu_a)} \to 0$. Let $\phi : \Omega \to V$ be defined as follows.

$$\forall \omega \in \Omega,\ \forall a \in (0,1), \quad \phi(\omega)(a) := \phi_{(a)}(\omega)$$

Clearly, $\phi_a = \phi_{(a)} \in M_\Omega$, for all $a \in (0,1)$. Hence $\phi \in M_{\Omega,V}$. On the other hand, since for each $a \in (0,1)$, $\phi_a = \phi_{(a)} \in L^{p(a)}(\Omega, \mu_a)$, we have $\phi \in L^p(\Omega, \mu)$. It remains to show that $d(\phi_n.\phi) \to 0$.

For $\epsilon > 0$, using (3.1) once more, there exists N such that $d(\phi_n, \phi_m) = \inf\{h \in (0,1) \mid \|\phi_n - \phi_m\|_p(1-h) < h\} < \epsilon$, for all $m, n \geq N$. Therefore, there is $h_0 < \epsilon$ with $\|\phi_n - \phi_m\|_p(1-h_0) < h_0$, from which it follows that $\|\phi_{n,c} - \phi_{m,c}\|_{L^{p(c)}(\Omega,\mu_c)} < h_0$, for all $c \in (0, 1-h_0)$. Using the fact that $\phi_{m,c} \to \phi_c$ in $L^{p(c)}(\Omega, \mu_c)$, we have

$$\|\phi_{n,c} - \phi_c\|_{L^{p(c)}(\Omega,\mu_c)} \leq h_0,$$

for all $c \in (0, 1-h_0)$ and $n \geq N$. Thus

$$\|\phi_n - \phi\|_p(1-h_0) = \sup_{c\in(0,1-h_0)} \|\phi_{n,c} - \phi_c\|_{L^{p(c)}(\Omega,\mu_c)} \leq h_0.$$

Hence for each $h \in (h_0, \epsilon)$,

$$\|\phi_n - \phi\|_p(1-h) \leq \|\phi_n - \phi\|_p(1-h_0) \leq h_0 < h < \epsilon$$

Therefore, $d(\phi_n, \phi) < \epsilon$, for all $n \geq N$. □

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