Probabilistic L^p Spaces

Farid Bahrami and Mehdi Mohammadbaghban*
Department of Mathematical Sciences, Isfahan University of Technology
Isfahan 84156, IRAN

December 28, 2011

Abstract

In this paper we define the notion of a probabilistic valued measure and the corresponding L^p spaces. With suitable interpretations, we also prove Hölder's inequality. These results give us sufficient motivations to define probabilistic dual spaces and probabilistic inner product spaces, for which the probabilistic L^2 space serves as a model.[1]

1 Introduction

Let D be the set of all distribution functions on real numbers, i.e. the set of all non-decreasing and left-continuous functions $F : \mathbb{R} \to [0, 1]$ with $\lim_{t\to-\infty} F(t) = 0$ and $\lim_{t\to+\infty} F(t) = 1$, and suppose $D^+ \subset D$ consists of all $F \in D$ with $F(0) = 0$. The notion of a probabilistic normed space (briefly, a PN-space) was first introduced by A.N.Šerstnev in [10], where he replaced the non-negative real numbers, as the target set of the norm function, with the elements of D^+. From then, many authors have investigated these spaces from different points of view; see for example [1], [3], [5], [7] and the text [8]. In this paper, following the idea of this replacement, we define the notion of a probabilistic valued measure, which is a generalization of finite measure, defined on a σ-algebra of subsets of a non-empty set, with values in D^+, instead of $[0, +\infty)$, and consider the problem of corresponding integral and L^p spaces. These latter spaces will turn out to constitute large classes of PN-spaces. With suitable interpretations for inequalities and product of distribution functions, we prove also the well-known Hölder's inequality. This will stimulate the question of probabilistic duals for L^p spaces. Also the special case $p = 2$ will serve as a good motivation to define probabilistic inner product spaces. However, these problems will be considered in another paper.

We first recall some definitions. For $r \in \mathbb{R}$, $\epsilon_r \in D$ is defined by

$$\forall t \in \mathbb{R}, \quad \epsilon_r(t) = \begin{cases} 0 & \text{if} \quad t \le r \\ 1 & \text{if} \quad t > r \end{cases}$$

*This work is partially supported by a grant from Isfahan University of Technology.

[1]2010 Mathematics Subject Classification: 60A10, 46B09.

Key words and phrases: probabilistic normed space, probabilistic valued measure, probabilistic L^p spaces, Hölder's inequality.

A partial ordering, denoted by $\leq$, is defined on D using the inverse partial ordering of real valued functions, i.e. for F and G in D, we say $F \leq G$ if for all $t \in \mathbb{R}$, $F(t) \geq G(t)$. With this convention, an element $F \in D$ belongs to D^+ if and only if $\epsilon_0 \leq F$.

Let S be a non-empty subset of D which is bounded from above, i.e. there exists $F \in D$ with $G \leq F$, for all $G \in S$. If $F_0 : \mathbb{R} \to [0,1]$ is defined by $F_0(t) = \inf_{G\in S} G(t)$ then F_0 is non-decreasing with $\lim_{t\to-\infty} F_0(t) = 0$ and $\lim_{t\to+\infty} F_0(t) = 1$. If we redefine F_0 at its points of discontinuity to make it lower semi-continuous, then $F_0 \in D$ and it will be the least upper bound of S in the partially ordered set $(D, \leq)$. Hence every subset of D which is bounded above has a least upper bound which, as in the classical case, we denote it by $\sup S$. Similarly, every subset of D which is bounded from below has an inf.

The set D can be made into a metric space with the metric $d_L : D \times D \to \mathbb{R}$, known as Levy metric, defined by

$$d_L(F,G) = \inf\{h \in [0,1] \mid F(t-h) - h \leq G(t) \leq F(t+h) + h \ \text{ for all } \ t \in \mathbb{R}\},$$

under which a sequence of distribution functions $(F_n)_{n\in\mathbb{N}}$ converges to $F \in D$ if and only if at each point $t \in \mathbb{R}$, where F is continuous, $F_n(t) \to F(t)$. Moreover, the metric space (D, d_L) is complete and separable [6].

A triangle function is a binary operation $\tau : D^+ \times D^+ \to D^+$ that is commutative, associative, non-decreasing in each variable and has ϵ_0 as identity. The most important triangle functions are those obtained from t-norms. A map $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it is commutative, associative, non-decreasing in each variable and with 1 as identity. An important example of a t-norm is M, defined by $M(u,v) = \min\{u,v\}$. As we will see in the next section, this t-norm plays a crucial role in our constructions. Corresponding to a t-norm T, there is defined a triangle function τ_T given by

$$\tau_T(F,G)(t) = \sup\{T\big(F(x), G(y)\big) \mid x + y = t\} \tag{1}$$

for all F and G in D^+ and $t \in \mathbb{R}$.

For a distribution function $F \in D$ and a non-negative $\lambda \in \mathbb{R}$, the multiplication of F by λ, denoted by λF, is defined by

$$(\lambda F)(t) = \begin{cases} \epsilon_0 & \text{if} \quad \lambda = 0 \\ F(\frac{t}{\lambda}) & \text{if} \quad \lambda > 0 \end{cases} \tag{2}$$

Clearly $\lambda F \in D$.

We are now ready to recall the definition of a PN-space in the sense of Šerstnev.

Definition 1.1 *Let X be a real vector space and τ be a continuous triangle function. A map $\nu : X \to D^+$ is called a probabilistic norm on X if for all $x, y \in X$ and $\lambda \in \mathbb{R}$ it satisfies the following conditions.*

(i) $\nu_x = \epsilon_0$ *if and only if* $x = 0$,

(ii) $\nu_{\lambda x} = |\lambda| \nu_x$,

(iii) $\nu_{x+y} \leq \tau(\nu_x, \nu_y)$,

where by $\nu_x \in D^+$ *we mean the value of* ν *at* $x \in X$. *In this case, the triple* (X, ν, τ) *is called a PN-space.*

In the above definition, if τ is given by the t-norm M, i.e. if $\tau = \tau_M$, then the corresponding PN-space is simply denoted by (X, ν).

In 1993, Alsina, Schweizer and Sempi gave a new definition of a PN-space which generalizes that of Definition 1.1. In their generalization, a probabilistic normed space is a quadruple (X, ν, τ, τ^*), where X and τ are as above, τ^* is a continuous triangle function with $\tau \leq \tau^*$, and $\nu : X \to \Delta^+$ is a function which satisfies conditions (i) and (ii) and the following two conditions.

(iv) $\nu_{-x} = \nu_x$,

(v) $\tau^*(\nu_{\lambda x}, \nu_{(1-\lambda)x}) \leq \nu_x$ for all $\lambda \in [0, 1]$.

It is known that if $\tau^* = \tau_M$ and equality holds in (v) for all $x \in X$ and $\lambda \in [0, 1]$, then the new definition coincides with that of Šerstnev's [2]. We refer the reader to [9] for a survey on the history of PN-spaces.

In this paper we consider the notion of a PN-space still in the sense of Šerstnev. As we pointed out, our aim is to define a probabilistic valued measure and the corresponding L^p spaces, as interesting classes of PN-spaces. To this end, the organization of the paper is as follows. In the next section, we introduce a certain dense subset of D^+ which will be used to prove an important inequality on D^+. In section 3, we define a new metric on D, finer than the Levy metric, which will be translation invariant. We also show that, under this metric, D^+ is complete. In the last section, we first define a probabilistic valued measure. Then, using the dense subset of D^+ introduced in section 2, we construct the smallest metrisable topological vector space which contains D^+. This latter space is used to introduce the notion of probabilistic integral of a measurable function. Having defined the probabilistic integrable functions, probabilistic L^p spaces and probabilistic norm on them are then defined in a natural way. The Hölder's inequality is proved as the last theorem of this section. We end the paper by a remark which raises the problem of probabilistic dual spaces and a new definition for probabilistic inner product spaces, all justified by the results obtained here.

2 Certain Dense Subsets of D^+

In this section, we introduce certain dense subsets of D^+ which will be used in the reminder of this paper. We will also investigate some properties of these subsets.

Let τ_M be the triangle function on D^+ corresponding to the t-norm M. It is easily seen that for each $F \in D^+$ and $\alpha, \beta \in [0, +\infty)$,

$$\tau_M(\alpha F, \beta F) = (\alpha + \beta)F$$

where the products αF, βF and $(\alpha+\beta)F$ are defined by (2). As Proposition 2.1 shows, this property characterizes the t-norm M.

We first introduce some notations which will be used throughout this section. For a distribution function $F \in D^+$ and a real $a \in [0,1)$, let $F_a^-, F_a^+ \in [0,+\infty)$ be defined as follows.

$$F_a^- = \inf\{t \in \mathbb{R} \mid a \le F(t)\} \tag{3}$$
$$F_a^+ = \sup\{t \in \mathbb{R} \mid F(t) \le a\} \tag{4}$$

In the proof of the following proposition we use the fact that if $t_a := F_a^-$, for some $a \in [0,1)$, then $F(t_a) \le a$.

Proposition 2.1 *Let $T : [0,1] \times [0,1] \to [0,1]$ be a continuous t-norm. If for all $F \in D^+$ and $\alpha, \beta \in (0,+\infty)$, the triangle function τ_T satisfies the following equation,*

$$\tau_T(\alpha F, \beta F) = (\alpha + \beta)F, \tag{5}$$

then $T = M$.

Proof. For $F \in D^+$, we will show that $T(a,a) = a$ for all $a \in F(\mathbb{R}) \cap [0,1)$. Hence $T(a,a) = a$ for all $a \in [0,1]$. Using the fact that $T \le M$, for each t-norm T, it then follows that $T = M$.

Let $F \in D^+$. Using (1) and (2), for all $\alpha, \beta > 0$ and $t \in \mathbb{R}$ we have

$$\begin{aligned} F(\frac{t}{\alpha+\beta}) &= \sup\{T\big((\alpha F)(x), (\beta F)(y)\big) \mid x + y = t\} \\ &= \sup\{T\big(F(\frac{x}{\alpha}), F(\frac{y}{\beta})\big) \mid x + y = t\} \end{aligned}$$

or equivalently,

$$F(t) = \sup\{T\big(F(\frac{x}{\alpha}), F(\frac{y}{\beta})\big) \mid x + y = (\alpha+\beta)t\}$$

for all $t \in \mathbb{R}$. In particular, for $\alpha = \beta = 1$ we obtain

$$\begin{aligned} F(t) &= \sup\{T\big(F(x), F(y)\big) \mid x + y = 2t\} \\ &= \sup\{T\big(F(t-z), F(t+z)\big) \mid z \in \mathbb{R}\} \end{aligned}$$

Fix $a \in F(\mathbb{R}) \cap [0,1)$ and let $t_a := F_a^-$. We will consider two cases.

Case 1. $F(t_a) < a$. Since $a \in F(\mathbb{R})$, there exists $t_0 \in (t_a, +\infty)$ such that $F(t_0) = a$. Hence $F(t) = a$ for all $t \in (t_a, t_0)$. Let t be the midpoint of the interval (t_a, t_0). Then for $z \le t_a - t$,

$$T\big(F(t-z), F(t+z)\big) \le T\big(F(t-z), F(t_a)\big) \le T\big(1, F(t_a)\big) = F(t_a) < a$$

The same inequality is obtained if $z \ge t - t_a$. Hence

$$\begin{aligned} a = F(t) &= \sup\{T\big(F(t-z), F(t+z)\big) \mid z \in \mathbb{R}\} \\ &= \sup\{T\big(F(t-z), F(t+z)\big) \mid z \in (t_a - t, t - t_a) = (t - t_0, t_0 - t)\} \\ &\le T\big(F(t_0), F(t_0)\big) = T(a,a) \end{aligned}$$

Thus $T(a,a) = a$.

Case 2. $F(t_a) = a$. For an arbitrary $\varepsilon > 0$, if $z \le -\varepsilon$ then

$$T\big(F(t_a - z), F(t_a + z)\big) \le T\big(1, F(t_a + z)\big) = F(t_a + z) < F(t_a) = a$$

The same inequality is true for $z \geq \varepsilon$. Hence

$$a = F(t_a) = \sup\{T\big(F(t_a - z), F(t_a + z)\big) \mid -\varepsilon < z < \varepsilon\} \tag{6}$$

Now suppose $T(a,a) < a$. Since F is left-continuous at t_a and $F(t) < a = F(t_a)$ for all $t < t_a$, we can choose $b \in F(\mathbb{R}) \cap (0,1)$ with $T(a,a) < b < a$. Clearly, in this case, $t_b := F_b^- < t_a$. If $F(t_b) < b$ then, as in case 1, it follows that $T(b,b) = b$. Hence

$$b = T(b,b) \leq T(a,a)$$

which is a contradiction. Thus $F(t_b) = b$. Now for $\varepsilon := t_a - t_b$, using the same argument which leads to (6), we will have

$$b = F(t_b) = \sup\{T\big(F(t_b - z), F(t_b + z)\big) \mid -\varepsilon < z < \varepsilon\} \leq T\big(F(t_a), F(t_a)\big) = T(a,a),$$

which is again a contradiction. Hence $T(a,a) = a$.■

Since the relation (5) is essential for our results in this paper, for the reminder of the paper, we will restrict ourselves to the t-norm M and, for simplicity, use the notation $\oplus$ instead of τ_M. Hence for $F, G \in D$, $F \oplus G \in D$ denotes the distribution function defined for each $t \in \mathbb{R}$ by

$$F \oplus G(t) = \sup\{\min\{F(x), G(y)\} \mid x + y = t\}$$

In particular, for $F \in D$ and $r \in \mathbb{R}$, we have $F \oplus \epsilon_r(t) = F(t - r)$.

For $F \in D$ and $r, s \in \mathbb{R}$ with $r \leq s$, let $C_{r,s}(F) \in D$ be defined as follows.

$$\forall t \in \mathbb{R}, \qquad C_{r,s}(F)(t) = \begin{cases} 0 & \text{if} \quad t \leq r \\ F(t) & \text{if} \quad r < t \leq s \\ 1 & \text{if} \quad s < t \end{cases} \tag{7}$$

It is easily seen that, for $r, s, u \in \mathbb{R}$ with $r \leq s$, $C_{r,s}(F) \oplus \epsilon_u = C_{r+u,s+u}(F \oplus \epsilon_u)$.

Lemma 2.2 *For F and G in D, suppose there exist $r, s \in \mathbb{R}$ such that $F = C_{-\infty,r}(F)$ and $G = C_{s,+\infty}(G)$. If $F(r) \leq \lim_{t \to s^+} G(t)$ then*

$$F \oplus G = (F \oplus \epsilon_s)\chi_{(-\infty, r+s]} + (G \oplus \epsilon_r)\chi_{(r+s,+\infty)}$$

Proof. Note that the assumptions of the lemma assert that $F(t) = 1$ for all $t > r$, and that $G(t) = 0$ for all $t \leq s$. Now let $t \in \mathbb{R}$. We consider the following two cases.
Case 1. $t \leq r + s$. For $u \in \mathbb{R}$, if $G(u) > 0$ then by assumption, $u > s$. Hence $t - u < r$ and so $F(t - u) \leq F(r) \leq G(u)$. Therefore, since $G(u) = 0$ for all $u \leq s$, we have

$$\begin{aligned} F \oplus G(t) &= \sup\{\min\{F(t-u), G(u)\} \mid u \in \mathbb{R}\} \\ &= \sup\{\min\{F(t-u), G(u)\} \mid s < u\} \\ &= \sup\{F(t-u) \mid s < u\} \\ &= F(t-s) = F \oplus \epsilon_s(t) \end{aligned}$$

Case 2. $r+s<t$. For $u\in\mathbb{R}$, $s<t-u$ if and only if $u<r+v$ where $v=t-(r+s)>0$. Moreover, $F(r)\le\lim_{t\to s^+}G(t)\le G(s+v)=G(t-r)$. Hence

$$\begin{array}{rcl} F\oplus G(t) & = & \sup\{\min\{F(u),G(t-u)\}\mid s<t-u\} \\ & = & \sup\{\min\{F(u),G(t-u)\}\mid u<r+v\} \\ & = & \sup\big(\{\min\{F(u),G(t-u)\}\mid u\le r\}\cup\{\min\{F(u),G(t-u)\}\mid r<u<r+v\}\big) \end{array}$$

If $u\le r$ then $t-u>s$. Hence $F(u)\le F(r)\le G(t-u)$. On the other hand, if $r<u$ then $F(u)=1$. Hence $\min\{F(u),G(t-u)\}=G(t-u)$. Thus

$$\begin{array}{rcl} F\oplus G(t) & = & \sup\big(\{F(u)\mid u\le r\}\cup\{G(t-u)\mid r<u<r+v\}\big) \\ & = & \max\{F(r),G(t-r)\} \\ & = & G(t-r)=G\oplus\epsilon_r(t) \end{array}$$

Therefore, for each $t\in\mathbb{R}$ we have proved that

$$F\oplus G(t)=\begin{cases} F\oplus\epsilon_s(t) & t\le r+s \\ G\oplus\epsilon_r(t) & t>r+s \end{cases}$$

Thus $F\oplus G=(F\oplus\epsilon_s)\chi_{(-\infty,r+s]}+(G\oplus\epsilon_r)\chi_{(r+s,+\infty)}$. ■

For $a\in[0,1)$, let $\gamma_a\in D^+$ be defined as follows.

$$\forall t\in\mathbb{R}\quad \gamma_a(t)=\begin{cases} 0 & t\le 0 \\ a & 0<t\le 1 \\ 1 & 1<t \end{cases} = a\chi_{(0,1]}(t)+\chi_{(1,+\infty)}(t) \tag{8}$$

Here χ_C denotes the characteristic function of a set C. Clearly $\gamma_0=\epsilon_1$. Moreover, for a non-negative $\lambda\in\mathbb{R}$, using (2), $\lambda\gamma_a$ is given by $a\chi_{(0,\lambda]}+\chi_{(\lambda,+\infty)}$. For a subset A of $[0,1)$, let $D^+_A\subset D^+$ be defined as the following set of distribution functions.

$$D^+_A=\{\lambda_1\gamma_{a_1}\oplus\cdots\oplus\lambda_n\gamma_{a_n}\mid n\in\mathbb{N},\ \forall i=1,\ldots,n,\ \lambda_i\ge 0,\ \text{and } a_i\in A\}$$

Elements of D^+_A can be written in another form. For $F=\lambda_1\gamma_{a_1}\oplus\cdots\oplus\lambda_n\gamma_{a_n}\in D^+_A$ with $a_1<\cdots<a_n$, using Lemma 2.2 and induction on n, it is easily seen that

$$F=a_1\chi_{(0,r_1]}+a_2\chi_{(r_1,r_2]}+\cdots+a_n\chi_{(r_{n-1},r_n]}+\chi_{(r_n,+\infty)}$$

where $r_i:=\lambda_1+\cdots+\lambda_i$ for $i=1,\ldots,n$. According to this form, for $F\in D^+_A$ and each $a\in[0,1)$, if F^-_a and F^+_a are defined as in (3) and (4) then, it is easily seen that $F^-_a<F^+_a$, if $a\in F(\mathbb{R})$, and $F^-_a=F^+_a$, otherwise. Moreover,

$$F=\sum_{a\in A}a\chi_{(F^-_a,F^+_a]}+\chi_{(F^-_1,+\infty)}.$$

Note that $(F^-_a,F^+_a]=\emptyset$ if $a\notin F(\mathbb{R})$. To take benefit from this representation of elements of D^+_A, we need some lemmas.

Lemma 2.3 *For $F, G \in D^+$ and $a \in [0,1)$, we have*

$$(F \oplus G)_a^- = F_a^- + G_a^- \qquad , \qquad (F \oplus G)_a^+ = F_a^+ + G_a^+$$

Proof. We prove the first identity. The second one is obtained similarly. For $t \in \mathbb{R}$ with $t > F_a^- + G_a^-$, we choose $t_1 > F_a^-$ and $t_2 > G_a^-$ such that $t = t_1 + t_2$. Then

$$\begin{aligned} (F \oplus G)(t) &= \sup\{\min\{F(t_1+x), G(t_2-x)\} \mid x \in \mathbb{R}\} \\ &\geq \min\{F(t_1), G(t_2)\} \geq a \end{aligned}$$

Hence $t \geq (F \oplus G)_a^-$. Since this is true for each $t \in \mathbb{R}$ with $t > F_a^- + G_a^-$, we obtain the inequality $F_a^- + G_a^- \geq (F \oplus G)_a^-$.

In order to prove the reverse inequality, we choose $t \in \mathbb{R}$, this times, with $t < F_a^- + G_a^-$, and $t_1, t_2 \in \mathbb{R}$ such that $t = t_1 + t_2$ and $t_1 < F_a^-$ and $t_2 < G_a^-$. Then $F(t_1) < a$ and $G(t_2) < a$. Now if $b < a$ is chosen such that $F(t_1) < b$ and $G(t_2) < b$, then

$$\min\{F(t_1+x), G(t_2-x)\} < b$$

for each $x \in \mathbb{R}$, and therefore

$$(F \oplus G)(t) = \sup\{\min\{F(t_1+x), G(t_2-x)\} \mid x \in \mathbb{R}\} \leq b < a$$

Hence $t < (F \oplus G)_a^-$. Thus, we obtain the inequality $F_a^- + G_a^- \leq (F \oplus G)_a^-$. ■

Using Lemmas 2.2 and 2.3, we obtain the following corollary.

Corollary 2.4 *For $F = \sum_{a\in A} a\chi_{(F_a^-, F_a^+]} + \chi_{(F_1^-, +\infty)}$ and $G = \sum_{a\in A} a\chi_{(G_a^-, G_a^+]} + \chi_{(G_1^-, +\infty)}$ in D_A^+, we have*

$$F \oplus G = \sum_{a\in A} a\chi_{(F_a^- + G_a^-, F_a^+ + G_a^+]} + \chi_{(F_1^- + G_1^-, +\infty)}$$

It is straight forward to see that, for $F = \sum_{a\in A} a\chi_{(F_a^-, F_a^+]} + \chi_{(F_1^-, +\infty)} \in D_A^+$ and a non-negative $\lambda \in \mathbb{R}$, λF as defined in (2), equals

$$\lambda F = \sum_{a\in A} a\chi_{(\lambda F_a^-, \lambda F_a^+]} + \chi_{(\lambda F_1^-, +\infty)}$$

Lemma 2.5 *For $F, G \in D^+$, $F \leq G$ if and only if $F_a^+ \leq G_a^+$, for all $a \in [0,1)$.*

Proof. First suppose $F \leq G$. For $a \in [0,1)$ and $t \in \mathbb{R}$ with $t < F_a^+$, we have $G(t) \leq F(t) \leq a$. Hence $t \leq G_a^+$. Since this true for all $t < F_a^+$, we get the inequality $F_a^+ \leq G_a^+$.

Conversely, suppose $F_a^+ \leq G_a^+$, for all $a \in [0,1)$. If $F \nleq G$ then there exists $t \in \mathbb{R}$ with $F(t) < G(t)$. By left-continuity of G, there is $s < t$ such that $F(t) < G(s) \leq G(t)$. Now if $a := F(t)$ then $F_a^+ \leq G_a^+ \leq s < t$, which contradicts the fact that $t \leq F_a^+$. ■

For a subset A of $[0,1)$, the same argument shows that for $F, G \in D_A^+$, $F \leq G$ if and only if $F_a^+ \leq G_a^+$, for all $a \in A$.

Remark 2.6 Suppose $F, G \in D$ and let

$$I_{F,G} := \{r \in [0,1] \mid F(t-r) - r \le G(t) \le F(t+r) + r, \text{ for all } t \in \mathbb{R}\}$$

It is clear that $1 \in I_{F,G}$. If $r_0 < 1$ belongs to $I_{F,G}$ then for $r \in (r_0, 1)$,

$$F(t-r) - r < F(t-r_0) - r_0 \le G(t) \le F(t+r_0) + r_0 < F(t+r) + r$$

for all $t \in \mathbb{R}$, i.e. $r \in I_{F,G}$. Hence, for $F, G \in D$, $I_{F,G}$ either equals the singleton $\{1\}$ or is a subinterval of $[0,1]$ containing 1. Moreover, it is easily seen that

$$I_{F,G} = \{r \in [0,1] \mid F(t) \le G(t+r) + r \text{ and } G(t) \le F(t+r) + r \text{ for all } t \in \mathbb{R}\}$$

Note that, according to the definition of Levy metric, $d_L(F,G) = \inf I_{F,G}$.

The following theorem demonstrates the importance of D_A^+ for certain subsets A of $[0,1)$.

Theorem 2.7 *Let A be a dense subset of $[0,1)$. Then the closure of D_A^+, in the metric space (D, d_L), equals D^+.*

Proof. Let $F \in D^+$ and $r > 0$. Choose $a_1, \ldots, a_n \in A$ with $a_0 := 0 < a_1 < \cdots < a_n < 1$ and such that $a_i - a_{i-1} < r$ for each $i = 1, \ldots, n$ and $1 - a_n < r$. Let $s_i := F_{a_i}^+$. Note that $F(s_i) \le a_i$ and $F(t) > a_i$ for $t > s_i$. Suppose $G \in D_A^+$ is given by $G = a_1\chi_{(0,s_1]} + \cdots + a_n\chi_{(s_{n-1},s_n]} + \chi_{(s_n,+\infty)}$. For $t \in \mathbb{R}$, if $t > s_n$ then

$$G(t) = 1 = 1 - r + r < a_n + r < F(t+r) + r \quad \text{and} \quad F(t) < 1 + r = G(t+r) + r.$$

If $t \in (s_{i-1}, s_i]$, for some $i = 1, \ldots, n$ then,

$$G(t) = a_i < a_{i-1} + r \le F(t+r) + r \quad \text{and} \quad F(t) \le a_i = G(t) \le G(t+r) + r$$

If $t \le 0$ then $F(t) = G(t) = 0$. Hence $d_L(F,G) \le r$, i.e. $D^+ \subset \overline{D_A^+}$. Inclusion in the reverse direction is clear. ■

Before closing this section, we show how certain functions $\phi : [0,+\infty) \to [0,+\infty)$ can be extended to D_A^+. Let $\mathcal{B}$ denote the σ-algebra of all Borel measurable subsets of $\mathbb{R}$ and suppose $\mathcal{M}$ denotes the set of all Borel probability measures on $\mathbb{R}$. If $\phi : \mathbb{R} \to \mathbb{R}$ is a measurable function (with respect to the measurable space $(\mathbb{R}, \mathcal{B})$) then for each $\mu \in \mathcal{M}$, the set function $\phi^*(\mu) : \mathcal{B} \to [0,+\infty)$, defined by $\phi^*(\mu)(B) = \mu(\phi^{-1}(B))$, belongs to $\mathcal{M}$. There is a metrizable topology on $\mathcal{M}$ under which a sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to $\mu \in \mathcal{M}$ if and only if $\int_{\mathbb{R}} h\,d\mu_n \to \int_{\mathbb{R}} h\,d\mu$ for all bounded continuous functions $h : \mathbb{R} \to \mathbb{R}$. Hence if $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function then, since $\int_{\mathbb{R}} h\,d\phi^*(\mu) = \int_{\mathbb{R}} h \circ \phi\,d\mu$, the map $\phi^* : \mathcal{M} \to \mathcal{M}$, given by $\mu \mapsto \phi^*(\mu)$, is continuous. There is also a one-to-one correspondence between $\mathcal{M}$ and D given by $\mu \mapsto F$, where $F : \mathbb{R} \to [0,1]$ is the distribution function defined by $F(t) = \mu(-\infty, t)$. This map is a homeomorphism if D is equipped with the metric d_L. Hence, if $\phi : \mathbb{R} \to \mathbb{R}$ is continuous then ϕ^* can be regarded as a continuous function on (D, d_L) (see [6]). By the same argument, if $\phi : [0,+\infty) \to [0,+\infty)$ is continuous then $\phi^* : D^+ \to D^+$ is also continuous.

Lemma 2.8 *Suppose $\phi : [0,+\infty) \to [0,+\infty)$ is a non-decreasing continuous function. For a subset A of $[0,1)$, ϕ^* is a continuous map on D_A^+, which is given for each $F = \sum_{a\in A} a\chi_{(F_a^-,F_a^+]} + \chi_{(F_1^-,+\infty)} \in D_A^+$ by*

$$\phi^*(F) = \sum_{a\in A} a\chi_{(\phi(F_a^-),\phi(F_a^+)]} + \chi_{(\phi(F_1^-),+\infty)} \tag{9}$$

Proof. Let $F \in D_A^+$. Then there exist $a_1,\ldots,a_n \in A$ with $a_1 < \cdots < a_n$ such that

$$F = a_1\chi_{(F_{a_1}^-,F_{a_1}^+]} + \cdots + a_n\chi_{(F_{a_n}^-,F_{a_n}^+]} + \chi_{(F_1^-,+\infty)}$$

The corresponding measure in $\mathcal{M}$ is in the form

$$\mu = a_1\delta_{F_{a_1}^-} + (a_2 - a_1)\delta_{F_{a_2}^-} + \cdots + (1 - \sum_{i=1}^n a_i)\delta_{F_1^-}$$

where δ_t denotes the dirac measure at $t \in \mathbb{R}$. Using the definition of $\phi^* : \mathcal{M} \to \mathcal{M}$, we have

$$\phi^*(\mu) = a_1\delta_{\phi(F_{a_1}^-)} + (a_2 - a_1)\delta_{\phi(F_{a_2}^-)} + \cdots + (1 - \sum_{i=1}^n a_i)\delta_{\phi(F_1^-)}$$

But $\phi(F_{a_1}^-) \le \cdots \le \phi(F_1^-)$, from which (9) is obtained. ■

If $\phi : [0,+\infty) \to [0,+\infty)$ is given by $\phi(t) = t^p$, for some $p > 0$, then for each $F \in D^+$ we demonstrate $\phi^*(F)$ simply by F^p.

Proposition 2.9 *Suppose $p \ge 1$. Then for $F_1,\ldots,F_n,G_1,\ldots,G_n \in D^+$ we have*

$$\Big((F_1 \oplus G_1)^p \oplus \cdots \oplus (F_n \oplus G_n)^p\Big)^{\frac{1}{p}} \le \Big(F_1^p \oplus \cdots \oplus F_n^p\Big)^{\frac{1}{p}} \oplus \Big(G_1^p \oplus \cdots \oplus G_n^p\Big)^{\frac{1}{p}} \tag{10}$$

Proof. According to Theorem 2.7, it suffices to prove this inequality on D_A^+. Hence we suppose $F_1,\ldots,F_n,G_1,\ldots,G_n \in D_A^+$. Using Lemmas 2.3 and 2.8, the left hand side of (10) can be written in the form

$$\sum_{a\in A} a\chi_{(s_a^-,s_a^+]} + \chi_{(s_1^-,+\infty)}$$

where $s_a^- = ((\sum_{i=1}^n (F_{i,a}^- + G_{i,a}^-)^p)^{\frac{1}{p}}$, $s_a^+ = (\sum_{i=1}^n (F_{i,a}^+ + G_{i,a}^+)^p)^{\frac{1}{p}}$ and $s_1^- = ((\sum_{i=1}^n (F_{i,1}^- + G_{i,1}^-)^p)^{\frac{1}{p}}$. Since

$$s_a^+ \le (\sum_{i=1}^n (F_{i,a}^+)^p)^{\frac{1}{p}} + (\sum_{i=1}^n (G_{i,a}^+)^p)^{\frac{1}{p}},$$

Lemma 2.5 leads to the result. ■

3 A new metric on the the set of distribution functions

In this section we construct a new metric on D. The priority of this metric over Levy metric will become clear as we demonstrate some of its properties.

In (4), F_a^+ was defined for $F \in D^+$ and $a \in [0,1)$. Here we extend this definition to distribution functions in D, and will denote it simply by F_a. Hence for $F \in D$ and $a \in (0,1)$,

$$F_a = \sup\{t \in \mathbb{R} \mid F(t) \leq a\}$$

As in Lemma 2.3, it is easily seen that $(F \oplus G)_a = F_a + G_a$ for all $F, G \in D$ and $a \in (0,1)$. Similarly, $(\lambda F)_a = \lambda F_a$, for $F \in D$ and $\lambda \geq 0$.

Lemma 3.1 *For $a, b \in (0,1)$ with $a < b$, let $d_{[a,b]} : D \times D \to [0, +\infty)$ be defined by*

$$d_{[a,b]}(F,G) = \sup\{|F_c - G_c| \mid c \in [a,b]\}$$

Then $d_{[a,b]}$ is a semi-metric on D. Moreover, for all $F, G, H \in D$ and $\lambda \geq 0$,

(i) $d_{[a,b]}(F,G) = d_{[a,b]}(F \oplus H, G \oplus H)$,

(ii) $d_{[a,b]}(\lambda F, \lambda G) = \lambda d_{[a.b]}(F,G)$.

Proof. The fact that $d_{[a,b]}$ is a semi-metric on D is observed readily. For $F, G, H \in D$, and $[a,b] \subset (0,1)$,

$$\begin{aligned} d_{[a,b]}(F,G) &= \sup\{|F_c - G_c| \mid c \in [a,b]\} \\ &= \sup\{|F_c + H_c - (G_c + H_c)| \mid c \in [a,b]\} \\ &= \sup\{((F \oplus H)_c - (G \oplus H)_c \mid c \in [a,b]\} \\ &= d_{[a,b]}(F \oplus H, G \oplus H) \end{aligned}$$

The other part is clear. ■

For $r > 0$, let $a_r := \frac{r}{2+2r}$ and $b_r := \frac{2+r}{2+2r}$. Then $0 < a_r < \frac{1}{2} < b_r < 1$ and $\lim a_r = \lim b_r = \frac{1}{2}$ as $r \to +\infty$. Hence for $F, G \in D$, one can always find $r > 0$ with $d_{[a_r,b_r]}(F,G) \leq r$.

Definition 3.2 *For $F, G \in D$, we define $d_P(F,G)$ as follows.*

$$d_P(F,G) = \inf\{r > 0 \mid d_{[a_r,b_r]}(F,G) \leq r\}$$

The following theorem states that d_P is a metric on D which is invariant under addition on this set. We omit the proof which is easily obtained from Lemma 3.1.

Theorem 3.3 *d_P is a metric on D which satisfies $d_P(F \oplus H, G \oplus H) = d_p(F,G)$, for all $F, G, H \in D$. Moreover, for $F, G \in D$ and $\lambda \geq 0$,*

$$d_p(F,G) \leq d_P(\lambda F, \lambda G) \leq \lambda d_P(F,G)$$

if $\lambda \geq 1$, and

$$d_p(F,G) \geq d_P(\lambda F, \lambda G) \geq \lambda d_P(F,G)$$

if $0 \leq \lambda < 1$.

We have seen in Theorem 2.7 that if A is a dense subset of $[0,1]$ then D_A^+ is dense in the metric space (D^+, d_L). The same is true if D^+ is equipped with the metric d_P. To show this property, we need the following observation. Let $F \in D^+$ and $a \in [0,1)$. For $r > 0$, if $b \in [0,1)$ is defined equal to $\inf\{c \in (a,1) \mid f_c - f_a \geq r\}$ then $a < b$ and $F_b - F_a \geq r$. Moreover, $F_c - F_a < r$ for each $c \in [a,b)$.

Theorem 3.4 *Let $A \subset [0,1)$ be a dense subset of $[0,1]$. Then D_A^+ is dense in the metric space (D^+, d_P).*

Proof. Let $F \in D^+$. For $r \in (0, \frac{1}{2})$, we define $G \in D^+$ equal to $C_{F_{a_r}, F_{b_r}}(F)$, as defined in (7). Then $d_{[a_r,b_r]}(F,G) = 0 \leq r$, and therefore $d_P(F,G) \leq r$. On the other hand, there exists a subset $\{c_0, c_1, \ldots, c_n\}$ of $[a_r, b_r]$ with $c_0 = a_r < c_1 < \cdots < c_n = b_r$ such that $F_c - F_{c_{i-1}} < r$ for all $c \in [c_{i-1}, c_i)$ and $i = 1, \ldots, n$. If $H \in D_A^+$ is defined equal to $c_1\chi_{(F_{c_0}, F_{c_1}]} + \cdots + c_n\chi_{(F_{c_{n-1}}, F_{c_n}]} + \chi_{(F_{c_n}, +\infty)}$ then, for $c \in [c_{i-1}, c_i]$,

$$|H_c - G_c| = |F_{c_{i-1}} - G_c| = |F_{c_{i-1}} - F_c| < r$$

Hence $d_P(G,H) \leq r$. Therefore $d_P(F,H) \leq 2r$. ■

Another important feature of the metric d_P is the completeness of D^+ under it, i.e. every Cauchy sequence in this metric space converges. In order to prove this property, we need the following lemma.

Lemma 3.5 *Let $\alpha : (0,1) \to [0, +\infty)$ be a non-decreasing and right-continuous function. Then there is a unique distribution function $F \in D^+$ with $F_a = \alpha(a)$, for all $a \in (0,1)$.*

Proof. We define $F : \mathbb{R} \to [0,1]$ as follows.

$$F(t) = \begin{cases} \inf\{c \in (0,1) \mid t \leq \alpha(c)\} & \text{if } \exists c \in (0,1) \text{ with } t \leq \alpha(c) \\ 1 & \text{otherwise} \end{cases}$$

It is easily verified that $F \in D^+$.

For $a \in (0,1)$, we choose $t \in \mathbb{R}$ with $\alpha(a) < t$. By right-continuity of α, there exists $c > a$ for which $\alpha(c) < t$. Thus $a < c \leq F(t)$, from which it follows that $F_a < t$. Since this is true for every $t > \alpha(a)$, we have $F_a \leq \alpha(a)$. To prove the reverse inequality, this time we choose $t \in \mathbb{R}$ such that $t > F_a$. Hence $a < F(t)$, and therefore $\alpha(a) < t$. Thus $\alpha(a) \leq F_a$. The uniqueness part is proved easily. ■

Proposition 3.6 *The metric space (D^+, d_P) is complete.*

Proof. Let $(F_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in (D^+, d_P). For a compact subinterval $[a,b]$ of $(0,1)$, we first choose $r_1 > 0$ with $a_{r_1} = \frac{r_1}{2+2r_1} < a$ and $b_{r_1} = \frac{2+r_1}{2+2r_1} > b$. Now, for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d_p(F_n, F_m) < \min\{\epsilon, r_1\}$, for all $m, n \geq N$. Hence, for these values of $m, n \in \mathbb{N}$,

$$\sup\{|F_{n,c} - F_{m,c}| \mid c \in [a,b]\} = d_{[a,b]}(F_n, F_m) \leq d_{[a_{r_1}, b_{r_1}]}(F_n, F_m) < \epsilon,$$

i.e. the sequence $(F_{n,c})_{n\in\mathbb{N}}$ is uniformly Cauchy on each subinterval $[a,b] \subset (0,1)$. If $\alpha : (0,1) \to \mathbb{R}$ is defined by $\alpha(c) = \lim_{n\to+\infty} F_{n,c}$, then α is non-decreasing and non-negative. On the other hand, for a distribution function $G \in D^+$, one can see that the function $c \mapsto G_c$ is right-continuous on $(0,1)$. Hence, being the uniform limit of a sequence of right-continuous functions on each compact subinterval of $(0,1)$, α is right-continuous on this interval. By Lemma 3.5, there exists $F \in D^+$ for which $F_c = \alpha(c)$, for all $c \in (0,1)$. Since $(F_{n,c})_{n\in\mathbb{N}}$ converges to F_c, uniformly for c in each compact subinterval of $(0,1)$, it follows that $d_P(F_n, F) \to 0$. ■

Combining Theorem 3.4 and Proposition 3.6, if A is a dense subset of $[0,1]$, then the metric space (D^+, d_P) is the completion of metric subspace (D^+_A, d_P), a fact which plays a crucial rule in the next section.

4 Probabilistic L^p spaces

In this section, we first define the notion of a probabilistic valued measure, which is a natural extension of finite measures, and then the corresponding L^p spaces. The probabilistic valued measures can easily be defined by replacing the set $[0,+\infty)$, the target space of a finite measure, with D^+. To do this we need the following lemma.

Lemma 4.1 *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence in the metric space (D^+, d_P) such that $\sum_{n=1}^{+\infty} F_n$ converges to some $F \in D^+$. Then for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{n=1}^{+\infty} F_{\sigma(n)}$ converges and has the same limit.*

Proof. Let $H_n := \sum_{i=1}^{n} F_i$, where $\sum_{i=1}^{n} F_i$ is a short form for $F_1 \oplus \cdots \oplus F_n$. Then, by assumption, for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_P(H_n, F) < \epsilon$, for all $n \geq N$. Let $M := \max\{\sigma^{-1}(1), \dots, \sigma^{-1}(N)\}$. Then for any $m \geq M$, since

$$H_N = \sum_{i=1}^{N} F_i \leq \sum_{i=1}^{m} F_{\sigma(i)} \leq F$$

we have

$$d_P(\sum_{i=1}^{m} F_{\sigma(i)}, F) \leq d_P(H_N, F) < \epsilon$$

which is the desired result. ■

Definition 4.2 *Let Ω be a non-empty set and Σ be a σ-algebra of its subsets. A set function $\mu : \Sigma \to D^+$ is called a probabilistic valued measure if for any mutually disjoint family of measurable subsets $\{A_n \mid n \in \mathbb{N}\}$, we have $\mu\big(\cup_{n\in\mathbb{N}} A_n\big) = \sum_{n\in\mathbb{N}} \mu(A_n)$.*

According to Lemma 4.1, the σ-additivity condition of the set function μ, is well-defined. In order to have an example, let $\mu : \Sigma \to [0,+\infty)$ be a finite measure. Then, for any $F \in D^+$, the set function $\mu_F : \Sigma \to D^+$, given by $\mu_F(A) = \mu(A)F$, is a probabilistic valued measure.

As in the classical case, it is easily seen that for a sequence $(A_n)_{n\in\mathbb{N}}$ of measurable subsets with $A_n \subseteq A_{n+1}$, for all $n \in \mathbb{N}$, if $A := \cup_{n\in\mathbb{N}} A_n$ then $\mu(A) = \lim \mu(A_n)$, in the metric space (D^+, d_P).

The main difficulty arises when one tries to define the corresponding integrable functions and their integrals. To overcome this problem, we first construct the smallest metrizable topological vector space, and in fact, the smallest PN-space, which contains a copy of the metric space (D^+, d_P).

Let A be a dense subset of $[0, 1)$ and suppose $c_{00}(A)$ denotes the set of all real valued functions with finite support, i.e. a function $f : A \to \mathbb{R}$ belong to $c_{00}(A)$ if there exists a finite set $A_f \subset A$ such that $f(a) = 0$ for all $a \in A \setminus A_f$. We denote by $c_{00}^+(A)$ the positive cone of non-negative elements in $c_{00}(A)$. Each $f \in c_{00}(A)$ can be decomposed in the form $f^+ - f^-$, where $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. Clearly $-f^- \le f \le f^+$. Moreover, $f^+, f^- \in c_{00}^+(A)$ are minimal with respect to this property, i.e. if $-g \le f \le h$, for some $g, h \in c_{00}^+(A)$, then $f^- \le g$ and $f^+ \le h$.

There is a natural map $\gamma : c_{00}^+(A) \to D_A^+$ defined for each $f \in c_{00}^+(A)$ by

$$\gamma(f) = \sum_{a\in A} f(a)\gamma_a$$

where $\gamma_a \in D_A^+$ is given by (8). It is easily seen that γ is one-to-one and on-to. Moreover, $\gamma(f + g) = \gamma(f) \oplus \gamma(g)$ and $\gamma(\lambda f) = \lambda\ \gamma(f)$, for all $f, g \in c_{00}^+(A)$ and $\lambda \ge 0$. Under the one-to-one correspondence γ, we identify $f \in c_{00}^+(A)$ and $\gamma(f) \in D_A^+$.

Lemma 4.3 *The function $d : c_{00}(A) \times c_{00}(A) \to [0, +\infty)$ defined by*

$$d(f, g) = d_P(f^+ \oplus g^-, f^- \oplus g^+)$$

is a metric on $c_{00}(A)$. Moreover, if we consider the metric spaces $\big(c_{00}^+(A), d\big)$ and (D_A^+, d_P), then the map $\gamma : c_{00}^+(A) \to D_A^+$ is an isometry.

Proof. We prove the triangle inequality for d. For $f, g, h \in c_{00}(A)$, using Theorem 3.3, we have

$$\begin{aligned}
d(f,g) &= d_P(f^+ \oplus g^-, f^- \oplus g^+)\\
&= d_P(f^+ \oplus h^- \oplus g^-, f^- \oplus h^- \oplus g^+)\\
&\le d_P(f^+ \oplus h^- \oplus g^-, f^- \oplus h^+ \oplus g^-) + d_P(f^- \oplus h^+ \oplus g^-, f^- \oplus h^- \oplus g^+)\\
&= d_P(f^+ \oplus h^-, f^- \oplus h^+) + d_P(h^+ \oplus g^-, h^- \oplus g^+)\\
&= d(f,h) + d(h,g)
\end{aligned}$$

The last assertion of the lemma is clear. In fact, for $f, g \in c_{00}^+(A)$,

$$d(f, g) = d_P(f^+ \oplus g^-, f^- \oplus g^+) = d_P(f, g)$$

■

The importance of the metric d becomes clear in the following theorem.

Theorem 4.4 *The metric space $\big(c_{00}(A), d\big)$ is a topological vector space.*

Proof. For $f_1, f_2, g_1, g_2 \in c_{00}(A)$, let $h_i := f_i^+ + g_i^+ - (f_i + g_i)^+ = f_i^- + g_i^- - (f_i + g_i)^-$, for $i = 1, 2$. Then $h_i \in c_{00}^+(A)$ and as elements of D_A^+, we have

$$f_i^+ \oplus g_i^+ = (f_i + g_i)^+ \oplus h_i \quad \text{and} \quad f_i^- \oplus g_i^- = (f_i + g_i)^- \oplus h_i$$

Thus

$$\begin{aligned}
d(f_1 + g_1, f_2 + g_2) &= d_P\big((f_1 + g_1)^+ \oplus (f_2 + g_2)^-, (f_1 + g_1)^- \oplus (f_2 + g_2)^+\big) \\
&= d_P\big((f_1 + g_1)^+ \oplus h_1 \oplus (f_2 + g_2)^- \oplus h_2, (f_1 + g_1)^- \oplus h_1 \oplus (f_2 + g_2)^+ \oplus h_2\big) \\
&= d_P(f_1^+ \oplus g_1^+ \oplus f_2^- \oplus g_2^-, f_1^- \oplus g_1^- \oplus f_2^+ \oplus g_2^+) \\
&\leq d_P(f_1^+ \oplus g_1^+ \oplus f_2^- \oplus g_2^-, f_1^- \oplus g_1^+ \oplus f_2^+ \oplus g_2^-) \\
&+ d_P(f_1^- \oplus g_1^+ \oplus f_2^+ \oplus g_2^-, f_1^- \oplus g_1^- \oplus f_2^+ \oplus g_2^+) \\
&= d_P(f_1^+ \oplus f_2^-, f_1^- \oplus f_2^+) + d_P(g_1^+ \oplus g_2^-, g_1^- \oplus g_2^+) \\
&= d(f_1, f_2) + d(g_1, g_2)
\end{aligned}$$

from which the continuity of addition follows. To prove the continuity of scalar multiplication, we first note that $d(f, g) = d(f-g, 0)$, from which it follows that $d(f+h, g+h) = d(f, g)$, for all $f, g, h \in c_{00}(A)$. Hence for $f, g \in c_{00}(A)$ and $\lambda, \mu \in \mathbb{R}$,

$$\begin{aligned}
d(\lambda f, \mu g) &= d\big((\lambda - \mu)f, \mu g - \mu f\big) \\
&\leq d\big((\lambda - \mu)f, 0\big) + d(0, \mu g - \mu f) \\
&= d\big((\lambda - \mu)f, 0\big) + d(\mu f, \mu g)
\end{aligned}$$

It is easily verified that $d(\mu f, \mu g) \leq \max\{1, |\mu|\} d(f, g)$ and $d\big((\lambda - \mu)f, 0\big) \leq |\lambda - \mu| t_0$, where $t_0 > 0$ is a real number for which the values of f^+ and f^-, as elements of D_A^+, at t_0 equal 1. Therefore, the scalar multiplication is also continuous. ■

Let $\big(\overline{c_{00}(A)}, d\big)$ be the completion of the topological vector space $\big(c_{00}(A), d\big)$. Since, by Lemma 4.3, the metric space (D_A^+, d_P) is isometrically embedded in $\big(c_{00}(A), d\big)$, according to the results obtained in the previous section, (D^+, d_P) is also isometrically embedded in $\big(\overline{c_{00}(A)}, d\big)$. Let $V \subset \overline{c_{00}(A)}$ be the smallest subspace containing D^+. Note that D^+ is a positive cone. Hence an element of V can be written in the form $F - G$, with $F, G \in D^+$. Having characterized the elements of V, one can easily define a probabilistic norm on this set. For an element $v \in V$, the set $\{F + G \mid F, G \in D^+,\ v = F - G\}$ in a non-empty subset of D^+, and therefore it has a greatest lower bound in this set. Now we define the absolute value of v, denoted by $\mathrm{abs}(v)$ by $\inf\{F + G \mid F, G \in D^+,\ v = F - G\}$. It is easily seen that (V, abs) is a PN-space. According to the definition of this probabilistic norm, it is clear that for $v = F - G \in V$, with $F, G \in D^+$, one has $\mathrm{abs}(v) \leq F + G$. Also, for $F \in D^+ \subset V$, $\mathrm{abs}(F) = F$. Note that in what follows, we do not use the topology obtained from this probabilistic norm on V.

We are now ready to define the integral corresponding to the probabilistic valued measure $\mu : \Sigma \to D^+$. Let $\phi : \Omega \to [0, +\infty)$ be a non-negative measurable simple function in the form $\sum_{i=1}^{k} a_i \chi_{A_i}$, where $\{A_i \mid i = 1, \ldots, k\}$ is a disjoint family of measurable subsets of Ω with $\Omega = \cup_{i=1}^{k} A_i$, and χ_{A_i} is the characteristic function of A_i. Then, we define

$$\int_\Omega \phi \, d\mu = a_1\mu(A_1) \oplus a_2\mu(A_2) \oplus \cdots \oplus a_n\mu(A_n),$$

which is denoted simply by $\sum_{i=1}^{k} a_i\mu(A_i)$. As in the classical case, it is easily seen that for two non-negative measurable simple functions ϕ and ψ, if $\phi \leq \psi$ then $\int_\Omega \phi \, d\mu \leq \int_\Omega \psi \, d\mu$, as two elements of D^+. We have also the additivity of the integral with respect to integrant, i.e. $\int_\Omega (\phi + \psi) \, d\mu = (\int_\Omega \phi \, d\mu) \oplus (\int_\Omega \psi \, d\mu)$. We omit the proof of the following lemma which is similar in spirit to the classical case, with D^+ in the place of $[0, +\infty)$. For its proof in the classical case one may refer, for example, to [4]. Recall that a non-empty subset of D^+ which is bounded above has always a least upper bound in D^+.

Lemma 4.5 *Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of non-negative measurable simple functions with $\phi_n \leq \phi_{n+1}$, for all $n \in \mathbb{N}$, which converges point-wise on Ω to some simple function ϕ. Then*

$$\int_\Omega \phi \, d\mu = \sup_n \int_\Omega \phi_n \, d\mu$$

Let $f : \Omega \to [0, +\infty)$ be a non-negative measurable function. We denote by S_f^+ the set of all measurable simple functions $\phi : \Omega \to [0, +\infty)$ with $\phi \leq f$.

Definition 4.6 *Let $f : \Omega \to [0, +\infty)$ be a measurable function. We say f is integrable with respect to μ, if there exists a distribution function $F \in D^+$ such that $\int_\Omega \phi \, d\mu \leq F$, for all $\phi \in S_f^+$, and in this case $\int_\Omega f \, d\mu$ is defined equal to $\sup\{\int_\Omega \phi \, d\mu \mid \phi \in S_f^+\}$.*

Lemma 4.7 *A measurable function $f : \Omega \to [0, +\infty)$ is integrable if and only if there exists $F \in D^+$ and a sequence $(\phi_n)_{n\in\mathbb{N}}$ in S_f^+ with*

$$\forall n \in \mathbb{N}, \quad \phi_n \leq \phi_{n+1} \qquad , \qquad \forall x \in \Omega, \quad \phi_n(x) \to f(x) \quad (\text{as } n \to +\infty) \tag{11}$$

such that $\int_\Omega \phi_n \, d\mu \leq F$ for all $n \in \mathbb{N}$, and in this case

$$\int_\Omega f \, d\mu = \sup \int_\Omega \phi_n \, d\mu. \tag{12}$$

Proof. If f is integrable then, by definition, there exists $F \in D^+$ which acts an upper bound for the set of distribution functions $\{\int_\Omega \phi \, d\mu \mid \phi \in S_f^+\}$. Therefore, for any sequence $(\phi_n)_{n\in\mathbb{N}}$ in S_f^+ which satisfies (11), we have $\int_\Omega \phi_n \, d\mu \leq F$. To prove (12) for this sequence, let $\phi \in S_f^+$ and $\psi_n := \min\{\phi, \phi_n\}$. Then $0 \leq \psi_n \leq \phi_n$, and $\psi_n \to \phi$, point-wise on Ω. Hence, by Lemma 4.5,

$$\int_\Omega \phi \, d\mu = \sup \int_\Omega \psi_n \, d\mu \leq \sup \int_\Omega \phi_n \, d\mu$$

Thus

$$\int_\Omega f \, d\mu = \sup_{\phi \in S_f^+} \int_\Omega \phi \, d\mu \leq \sup \int_\Omega \phi_n \, d\mu$$

The inequality in the reverse direction is clear.

Conversely, suppose $(\phi_n)_{n\in\mathbb{N}}$ is a sequence in S_f^+ which satisfies (11) and

$$\forall n \in \mathbb{N}, \quad \int_\Omega \phi_n \, d\mu \leq F$$

for some $F \in D^+$. As in the previous part, for any $\phi \in S_f^+$ we have

$$\int_\Omega \phi \, d\mu \leq \sup \int_\Omega \phi_n \, d\mu \leq F$$

Hence, by Definition 4.6, f is integrable. ■

Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of distribution functions in D^+ with $F_n \leq F_{n+1}$, for all $n \in \mathbb{N}$ which is also bounded above. Then for $F \in D^+$, it is easily seen that $F = \sup F_n$ if, and only if, $d_L(F_n, F) \to 0$.

Corollary 4.8 *If the measurable functions $f, g : \Omega \to [0, +\infty)$ are integrable then so is $f+g$ and $\int_\Omega (f+g)\, d\mu = (\int_\Omega f \, d\mu) \oplus (\int_\Omega g \, d\mu)$.*

Proof. By Lemma 4.7, there exist $F, G \in D^+$ and sequences $(\phi_n)_{n\in\mathbb{N}}$, in S_f^+, and $(\psi_n)_{n\in\mathbb{N}}$, in S_g^+, with

$$\forall n \in \mathbb{N},\ \phi_n \leq \phi_{n+1},\ \psi_n \leq \psi_{n+1} \quad , \quad \forall x \in \Omega,\ \phi_n(x) \to f(x),\ \phi_n(x) \to g(x)$$

and such that $\int_\Omega \phi_n \, d\mu \leq F$ and $\int_\Omega \psi_n \, d\mu \leq G$, for all $n \in \mathbb{N}$. Since $(\phi_n + \psi_n)_{n\in\mathbb{N}}$ is an increasing sequence in $S_{(f+g)}^+$ which converges point-wise to $f+g$, and

$$\forall n \in \mathbb{N}, \quad \int_\Omega (\phi_n + \psi_n)\, d\mu = \int_\Omega \phi_n \, d\mu \oplus \int_\Omega \psi_n \, d\mu \leq F \oplus G$$

Hence $f+g$ is integrable. Moreover, since the operation $\oplus : D^+ \times D^+ \to D^+$ is continuous, when D^+ is equipped with d_L, we have

$$\begin{aligned} \int_\Omega (f+g)\, d\mu = \sup_n \Big(\int_\Omega \phi_n \, d\mu \oplus \int_\Omega \psi_n \, d\mu\Big) &= \lim_n \Big(\int_\Omega \phi_n \, d\mu \oplus \int_\Omega \psi_n \, d\mu\Big) \\ &= \lim_n \int_\Omega \phi_n \, d\mu \oplus \lim_n \int_\Omega \psi_n \, d\mu \\ &= \int_\Omega f \, d\mu \oplus \int_\Omega g \, d\mu \end{aligned}$$

■

Having defined the integrability of a non-negative measurable function, the definition of L^p spaces follow automatically. For $p \geq 1$, the probabilistic L^p space with respect to the probabilistic valued measure $\mu : \Sigma \to D^+$, denoted by $L^p(\Omega, \mu)$, is defined equal to the set of all measurable functions $f : \Omega \to \mathbb{R}$ for which $|f|^p : \Omega \to [0, +\infty)$ is integrable. Clearly

$L^p(\Omega,\mu)$ is a linear space. In the case $p=1$, if $f\in L^1(\Omega,\mu)$ is written in the form f^+-f^- then

$$\int_\Omega f\,d\mu := \int_\Omega f^+\,d\mu - \int_\Omega f^-\,d\mu \in V$$

We have also the natural inequality

$$\mathrm{abs}(\int_\Omega f\,d\mu) \leq \int_\Omega |f|\,d\mu$$

Theorem 4.9 *The function $\|\cdot\|_p : L^p(\Omega,\mu)\to D^+$ defined by*

$$\|f\|_p = \Big(\int_\Omega |f|^p\,d\mu\Big)^{\frac{1}{p}}$$

satisfies the triangle inequality, i.e. for $f,g\in L^p(\Omega,\mu)$, we have

$$\|f+g\|_p \leq \|f\|_p + \|g\|_g \tag{13}$$

Proof. It suffices to prove the triangle inequality for non-negative measurable simple functions. So suppose $\phi,\psi:\Omega\to[0,+\infty)$ are two measurable simple functions in the form $\phi=\sum_{i=1}^n a_i\chi_{A_i}$ and $\psi=\sum_{i=1}^n b_i\chi_{A_i}$ with $\{A_i \mid i=1,\dots,n\}$ a family of disjoint measurable subsets of Ω such that $\Omega=\cup_{i=1}^n A_i$. Using Proposition 2.9, we have

$$\begin{aligned}
\int_\Omega(\phi+\psi)^p\,d\mu &= \sum_{i=1}^n (a_i+b_i)^p\mu(A_i)\\
&\leq \Big(\big(\sum_{i=1}^n a_i^p\mu(A_i)\big)^{\frac{1}{p}} \oplus \big(\sum_{i=1}^n b_i^p\mu(A_i)\big)^{\frac{1}{p}}\Big)^p\\
&= \Big(\big(\int_\Omega \phi^p\,d\mu\big)^{\frac{1}{p}} \oplus \big(\int_\Omega \psi^p\,d\mu\big)^{\frac{1}{p}}\Big)^p
\end{aligned}$$

■

Two measurable functions $f,g:\Omega\to\mathbb{R}$ are said equal almost every where on Ω if $\mu(\{x\in\Omega \mid f(x)\neq g(x)\})=\epsilon_0$. As in the classical case, this relation defines an equivalence relation on $L^p(\Omega,\mu)$, and if we denote the set of all equivalent classes still by $L^p(\Omega,\mu)$ then $\|\cdot\|_p : L^p(\Omega,\mu)\to D^+$ defines a probabilistic norm on this space.

The last property we consider is Hölder,s inequality. We first have to define an appropriate multiplication on D^+. As it was pointed out before, for $F\in D^+$, the map $a\mapsto F_a$ defines a non-decreasing, right-continuous and non-negative function on $(0,1)$. Hence for $F,G\in D^+$, the map $a\mapsto F_aG_a$, defined on $(0,1)$, satisfies the conditions of Lemma 3.5. Hence there exists a unique distribution function in D^+, which we denote it by FG, for which

$$\forall a\in(0,1), \qquad (FG)_a = F_aG_a.$$

Lemma 4.10 *For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $F_1, \ldots, F_n, G_1, \ldots, G_n \in D^+$, we have*

$$\sum_{i=1}^{n} F_i G_i \leq (\sum_{i=1}^{n} F_i^p)^{\frac{1}{p}} (\sum_{i=1}^{n} G_i^q)^{\frac{1}{q}}$$

Proof. Let H and K in D^+ be given by $H = F_1 G_1 \oplus \cdots \oplus F_n G_n$ and $K = (\sum_{i=1}^{n} F_i^p)^{\frac{1}{p}} (\sum_{i=1}^{n} G_i^q)^{\frac{1}{q}}$. For every $a \in (0,1)$, we have $H_a = F_{1,a} G_{1,a} + \cdots + F_{n,a} G_{n,a}$. On the other hand, for a distribution function $F \in D^+$ and $p \geq 1$, it is easily verified that $\big(F^p\big)_a = \big(F_a\big)^p$. Hence, for every $a \in (0,1)$,

$$\begin{aligned} K_a &= \Big((\sum_{i=1}^{n} F_i^p)^{\frac{1}{p}}\Big)_a \Big((\sum_{i=1}^{n} G_i^q)^{\frac{1}{q}}\Big)_a \\ &= (\sum_{i=1}^{n} F_{i,a}^p)^{\frac{1}{p}} (\sum_{i=1}^{n} G_{i,a}^q)^{\frac{1}{q}} \end{aligned}$$

Since

$$H_a = \sum_{i=1}^{n} F_{i,a} G_{i,a} \leq (\sum_{i=1}^{n} F_{i,a}^p)^{\frac{1}{p}} (\sum_{i=1}^{n} G_{i,a}^q)^{\frac{1}{q}} = K_a$$

for every $a \in (0,1)$, Lemma 2.5 leads to the desired inequality. ∎

We are now ready to state and prove Hölder,s inequality.

Theorem 4.11 *For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, if $f \in L^p(\Omega, \mu)$ and $g \in L^q(\Omega, \mu)$ then $fg \in L^1(\Omega, \mu)$ and*

$$\text{abs}(\int_\Omega fg \, d\mu) \leq \|f\|_p \, \|g\|_q$$

Proof. It suffices to prove Hölder's inequality for two non-negative simple measurable functions. Let $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ and $\psi = \sum_{i=1}^{n} b_i \chi_{A_i}$ be two non-negative measurable functions, with $\{A_i \mid i = 1, \ldots, n\}$ a disjoint family of measurable subsets of Ω. Then, by Lemma 4.10,

$$\begin{aligned} \int_\Omega \phi\psi \, d\mu = \sum_{i=1}^{n} a_i b_i \mu(A_i) &= \sum_{i=1}^{n} a_i \mu(A_i)^{\frac{1}{p}} b_i \mu(A_i)^{\frac{1}{q}} \\ &\leq \big(\sum_{i=1}^{n} a_i^p \mu(A_i)\big)^{\frac{1}{p}} \big(\sum_{i=1}^{n} b_i^q \mu(A_i)\big)^{\frac{1}{q}} \\ &= \|\phi\|_p \, \|\psi\|_q \end{aligned}$$

∎

Remark 4.12 Let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $g \in L^q(\Omega, \mu)$ then, by Theorem 4.11, the map $T_g : L^p(\Omega, \mu) \to V$, given by $T_g(f) = \int_\Omega fg \, d\mu$, is a well-defined linear map. Moreover, if V is equipped with the topology corresponding to the probabilistic norm abs : $V \to D^+$, then T_g is seen to be continuous. Comparing this situation with the classical case, it seems natural to speak of the probabilistic dual space, as the set of all linear continuous maps $T : L^p(\Omega, \mu) \to V$ and consider the problem of Riesz Representation Theorem. All these considerations makes it reasonable to pose the problem of substituting $\mathbb{R}$ with the PN-space (V, abs) and investigating the corresponding structures. One of these structures is the probabilistic inner product space, for which the probabilistic $L^2(\Omega, \mu)$ serves as a model. This will be the subject of a subsequent paper.

References

[1] Aghajani, A. and Nourouzi, K. *Convex sets in probabilistic normed spaces*, Chaos, Solitons and Fractals, 36 (2008) 322-328.

[2] Alsina,C. , Schweizer,B. and Sklar,A. , *On the definition of a probabilistic normed space*, Aequationes Math., 46 (1993), 91-98.

[3] Bahrami,F. and Nikfar,M. , *The topological structure of a certain Menger space*, J. Math. Anal. Appl. 334 (2007), 172-182.

[4] Cohn, D.L., Measure Theory, Birkhuser Boston, 1980.

[5] Fang, J.X., *On non-linear equations for fuzzy mappings in probabilistic normed spaces*, Fuzzy Sets and Systems, Vol. 131 (3) (2002), 357-364.

[6] Itô, K., Introduction to probability theory, Cambridge University Press, 1984.

[7] Mihet, D. and Radu, V., *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl., 343 (2008) 567-572.

[8] Schweizer,B. and Sklar,A., Probabilistic metric spaces, Elsevier North-Holland, New York, 1983.

[9] Sempi,C. , *A Short and Partial History of Probabilistic Normed Spaces*, Mediterr. J. Math., 3 (2006), 283-300.

[10] Šerstnev,A.N. , *On the notion of a random normed space*, Dokl. Akad. Nauk SSSR, 142 (2) (1963), 280-283.