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EXISTENCE OF A UNIFORM BOUND FOR QUADRATURE DOMAINS ASSOCIATED TO p-LAPLACIAN

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Abstract. In this paper[1], we investigate for a real $p > 1$, the notion of p-quadrature domains in $\mathbf{R}^n$. A uniqueness theorem for these domains has been recently obtained by Bahrami-Shahgholian under the form of an over determined boundary value problem. We prove here that if μ is an absolutely continuous measure with compact support and density in $L^\infty(\mathbf{R}^n)$, then all the bounded p-quadrature domains are included in a bounded domain Ω_0, depending only on μ and p.

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1 Introduction

Suppose $p > 1$ is a real number and Δ_p denotes the p-Laplacian defined by :

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u).$$

Given a Radon measure μ with compact support K in $\mathbf{R}^n$, a domain $\Omega \subset \mathbf{R}^n$ is called a *quadrature domain for μ with respect to the operator Δ_p* (or simply a *$p-QD$ for μ*) if $K \subset \Omega$

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and there exists a real valued function u in the Sobolev space $W^{1,p}_{loc}(\mathbf{R}^n)$ satisfying

$$(1) \qquad \begin{cases} \Delta_p u = \chi_\Omega - \mu & in \ \ \mathbf{R}^n \\ u = |\nabla u| = 0 & in \ \ \mathbf{R}^n \setminus \Omega \end{cases}$$

where χ_Ω denotes the characteristic function of Ω. The first equation in (1) is considered in the sense of distributions, i.e. the equality

$$\int_{\mathbf{R}^n} (|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi) dx + \int_\Omega \varphi dx = \int \varphi d\mu$$

holds for every $\varphi \in C_0^\infty(\mathbf{R}^n)$, where dx is the usual Lebesgue measure in $\mathbf{R}^n$. The set of all pairs (Ω, u) in which Ω is bounded and u satisfies (1) for a given measure μ, will be denoted by $QD_p(\mu)$.

Note that for $p = 2$, the p-Laplacian reduces to the ordinary Laplacian and the notion of a 2-QD coincides with that of the classical quadrature domains which have extensively been studied. We refer the reader to Sakai [6] and Gustafsson [3] and references therein for precise definitions and some properties of classical quadrature domains. We also refer to Gustafsson-Shahgholian [4] where this classical notion has been dealed with in the form of a *free boundary problem* similar to what we have introduced above for $p = 2$, and finally to Bahrami-Shahgholian [1] which may serve in fact as a starting point for the notion of p-QD.

The problems of existence, uniqueness and regularity of such domains are of interest and offer substantial challenges even in the classical case. However analogous problems for general p seem also to be interesting. Indeed recent applications of non-linear potential theory, developed around Δ_p, have led many authors to generalize the already existing notions in classical potential theory and to encounter them in a more general setting, suitable from the point of view of non-linear theory.

It is therefore quiet natural to study for p-QDs some important properties previously obtained for quadrature domains. The uniqueness problem has been partially answered by Bahrami-Shahgholian [1]. Their result extends an earlier theorem of Shahgholian [8] and says in the present terminology that if Ω_1 and Ω_2 are two bounded p-QDs for a Radon measure with compact support and continuous with respect to p-capacity, and if $\Omega_1 \cap \Omega_2$ is convex then $\Omega_1 = \Omega_2$.

In this paper we are concerned with the problem of existence of a uniform bound for bounded p-QDs, generalising the following:

Theorem (Shahgholian [7]): Let T be a distribution with compact support in $\mathbf{R}^n$. Then there is a bounded domain $\Omega_0 \subset \mathbf{R}^n$ containing all (if any) bounded domains $\Omega \subset \mathbf{R}^n$ for which the following over determined Cauchy problem

$$\begin{cases} \Delta u = 1 - T & in \quad \Omega \\ u = |\nabla u| = 0 & on \quad \partial\Omega \end{cases}$$

has a solution.

Our main theorem is as follows:

THEOREM 1. *Suppose μ is an absolutely continuous measure with compact support and density in $L^\infty(\mathbf{R}^n)$. If $QD_p(\mu)$ is non-empty, then there exists a fixed bounded domain $\Omega_0 \subset \mathbf{R}^n$ such that $\Omega \subset \Omega_0$ for all Ω in $QD_p(\mu)$.*

It must be pointed out that, due to the nonlinearity of Δ_p for $p \neq 2$, the techniques used here are completely different from those in [7]. Our approach is based on minimizing a special functional over a closed convex subset of the Sobolev space and on introducing an upper barrier for the minimizer. We will also make use of the comparison principle for which we refer the reader to [5].

2 Preliminaries

NOTATIONS. Throughout the paper B_R will denote the ball $\{x \in \mathbf{R}^n, |x| < R\}$. The measure μ is assumed to be an absolutely continuous measure with density in $L^\infty(\mathbf{R}^n)$ and with coompact support. For simplicity we will denote its density function by μ itself. The norm of the space L^p with respect to the Lebesgue measure, dx, is denoted by $\|.\|_p$. We also let $W_0^{1,p}(\Omega)$ to stand for the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ and will consider it with the corresponding norm $\|u\| = (\int_\Omega |\nabla u|^p dx)^{1/p}$. For a domain $D \subset \mathbf{R}^n$, the convex cone $\{v \in W_0^{1,p}(D), v \geq 0\}$ will be denoted by $\mathcal{K}(D)$. The class of differentiable real valued functions with continuous first order derivatives and with compact support in D is denoted by $C_0^1(D)$. For $1 < p < \infty$, we let $q \in \mathbf{R}$ denote the inverse conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

2.1 Minimizers

Suppose $D \subset \mathbf{R}^n$ is a bounded domain. Given a function $f \in L^q(D)$, let us denote by J_f the functional

$$J_f(v) = \int_D (\frac{1}{p}|\nabla v|^p + fv)dx \tag{2}$$

defined on $W^{1,p}(D)$. Using classical arguments, one obtains easily some results concerning minimizers for this functional over the closed convex cone $\mathcal{K}(D)$.

The first proposition states the existence and uniqueness of the minimizer.

PROPOSITION 2.1.1. *J_f has a unique minimizer over $\mathcal{K}(D)$.*□

We omit the proof which is based on the classical properties of coercivity and weak lower semicontinuity of J_f, and the fact that J_f is bounded below over $\mathcal{K}(D)$ (see [2]).

PROPOSITION 2.1.2. *If $u_0 \in C_0^1(D)$ is a non-negative function and $\Delta_p u_0 = f$ on D, then u_0 is the minimizer of J_f over $\mathcal{K}(D)$.*□

The proof is more or less straight forward and can be found for example in [5].

2.2 Barrier Function

In the next lemma we will introduce a function which will serve as an upper barrier for our minimizers.

LEMMA 2.2.1. *For given constants $r_0 > 0$ and $M > 1$, let*

$$a = (\frac{n}{M-1} + 1)(\frac{M-1}{n})^q r_0^q$$

(3)
$$R_0 = r_0(\frac{M-1}{n}+1)$$

Then the well defined function

$$u_0(x) = \begin{cases} \frac{a}{q} - \frac{1}{q}(\frac{M-1}{n})^{q-1}|x|^q & |x| \le r_0 \\ \frac{1}{q}(R_0 - |x|)^q & r_0 \le |x| \le R_0 \\ 0 & R_0 \le |x| \end{cases}$$

is in $C_0^1(\mathbf{R}^n)$ and satisfies

$$\Delta_p u_0 = (1-M)\chi_{B_{r_0}} + h\chi_{B_{R_0}\setminus B_{r_0}}$$

where $h(x) \le 1$ for $x \in B_{R_0} \setminus B_{r_0}$.

Proof: The property $u_0 \in C_0^1(\mathbf{R}^n)$ is verified easily. Moreover we have

$$(|\nabla u_0|^{p-2}\nabla u_0)(x) = \begin{cases} -(\frac{M-1}{n})x & |x| < r_0 \\ -\frac{R_0 - |x|}{|x|}x & r_0 < |x| < R_0 \\ 0 & R_0 < |x| \end{cases}$$

This expression can be extended continuously to $|x| = r_0$ using (3), and to $|x| = R_0$ using its very definition. Thus for $\varphi \in C_0^\infty(\mathbf{R}^n)$, using integration by parts, we have

(4)
$$\int_{\mathbf{R}^n} |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\varphi = -(\frac{M-1}{n})\int_{B_{r_0}} x\cdot\nabla\varphi - \int_{B_{R_0}\setminus B_{r_0}} \frac{R_0 - |x|}{|x|}x\cdot\nabla\varphi$$

$$= -\left(\frac{M-1}{n}\right)\Big(\int_{\partial B_{r_0}} \varphi(\zeta)\zeta\cdot\mathbf{n}_1(\zeta)d\zeta - \int_{B_{r_0}} (div\ x)\varphi\Big)$$
$$-\Big(\int_{\partial B_{R_0}} \varphi(\zeta)\frac{R_0-|\zeta|}{|\zeta|}\zeta\cdot\mathbf{n}_3(\zeta)d\zeta + \int_{\partial B_{r_0}} \varphi(\zeta)\frac{R_0-|\zeta|}{|\zeta|}\zeta\cdot\mathbf{n}_2(\zeta)d\zeta\Big)$$
$$-\int_{B_{R_0}\setminus B_{r_0}} div\Big(\frac{R_0-|x|}{|x|}x\Big)\varphi$$

where $d\zeta$ is the surface measure associated to the Lebesgue measure. Here $\mathbf{n}_1$ and $\mathbf{n}_2$ are respectively the outward and inward unit normals to ∂B_{r_0}. Similarly $\mathbf{n}_3$ denotes the outward unit normal to ∂B_{R_0}. Using (3) once more, we have

(5) $$\int_{\partial B_{r_0}} \varphi(\zeta)\frac{R_0-|\zeta|}{|\zeta|}\zeta\cdot\mathbf{n}_2(\zeta)d\zeta = -\frac{R_0-r_0}{r_0}\int_{\partial B_{r_0}} \varphi(\zeta)\zeta\cdot\mathbf{n}_1(\zeta)d\zeta$$
$$= -\frac{M-1}{n}\int_{\partial B_{r_0}} \varphi(\zeta)\zeta\cdot\mathbf{n}_1(\zeta)d\zeta$$

It is also clear that

(6) $$\int_{\partial B_{R_0}} \varphi(\zeta)\frac{R_0-|\zeta|}{|\zeta|}\zeta\cdot\mathbf{n}_3(\zeta)d\zeta = 0$$

Thus using (5) and (6) in (4), we obtain that

$$\int_{\mathbf{R}^n} |\nabla u_0|^{p-2}\nabla u_0\cdot\nabla\varphi = (M-1)\int_{B_{r_0}}\varphi + \int_{B_{R_0}\setminus B_{r_0}}\Big(\frac{R_0(n-1)}{|x|}-n\Big)\varphi$$

for all $\varphi\in C_0^\infty(\mathbf{R}^n)$, i.e.

$$-div(|\nabla u_0|^{p-2}\nabla u_0) = (M-1)\chi_{B_{r_0}} + \Big(\frac{R_0(n-1)}{|x|}-n\Big)\chi_{B_{R_0}\setminus B_{r_0}}$$

as desired.□

3 Main Result

Before proving the main theorem, we need a lemma, analogous to lemma 3 in [1], concerning a quadrature domain and its corresponding function.

LEMMA 3.1. *Suppose $(\Omega, u)\in QD_p(\mu)$. For any $\zeta\in\partial\Omega$ and any neighbourhood of ζ, such as $U_\zeta\subset\mathbf{R}^n$, there is a $z\in U_\zeta\cap\Omega$ such that $u(z)>0$.*

Proof : Suppose on the contrary that for some $\zeta\in\partial\Omega$ there is a nieghbourhood U_ζ such that $u\,|_{U_\zeta}\le 0$. Choose U_ζ small enough so that $U_\zeta\cap supp(\mu)=\emptyset$. Since $\Delta_p u\ge 0$ in U_ζ, it is p-subharmonic in U_ζ. According to [5] chapter 7, this contradicts the fact that u has attained its maximum 0 inside U_ζ.□

We are now ready to prove our main result.

Proof of theorem 1. Choose $r_0 > 0$ and $M > 1$ such that $supp(\mu) \subset B_{r_0}$ and $\|\mu\|_\infty < M$. By lemma 2.2.1, there is a function $u_0 \in C_0^1(\mathbf{R}^n)$ which satisfies

$$\Delta_p u_0 = 1 - M \quad \text{in } B_{r_0}$$

and

$$\Delta_p u_0 \leq 1 \quad \text{in } B_R \setminus B_{r_0}$$

for any $R > r_0$. Also note that $supp(u_0) = B_{R_0}$ where R_0 is as defined by (3).

Let $(\Omega, u) \in QD_p(\mu)$ and choose $R > R_0$ large enough so that $\Omega \subset B_R$. Consider the functional

$$J(v) = \int_{B_R} (\frac{1}{p}|\nabla v|^p + (1-\mu)v)dx$$

over $\mathcal{K}(B_R) = \{v \in W_0^{1,p}(B_R), v \geq 0\}$. By proposition 2.1.1, J has a unique minimizer, $\tilde{u}$, over $\mathcal{K}(B_R)$. We first show that $\tilde{u} \leq u_0$.

To do this, let $f = \Delta_p u_0$. By proposition 2.1.2, u_0 is the minimizer of the functional J_f (introduced as in section 2.1) over $\mathcal{K}(B_R)$. Let $v = min(\tilde{u}, u_0)$ and $w = max(\tilde{u}, u_0)$. Then

$$\int_{B_R} \frac{1}{p}|\nabla \tilde{u}|^p + \int_{B_R} \frac{1}{p}|\nabla u_0|^p = \int_{B_R} \frac{1}{p}|\nabla v|^p + \int_{B_R} \frac{1}{p}|\nabla w|^p \tag{7}$$

$$\begin{aligned} \int_{B_R} (1-\mu)\tilde{u} + \int_{B_R} f u_0 &= \int_{B_R} f(u_0 + \tilde{u}) + (1 - \mu - f)\tilde{u} \\ &= \int_{B_R} f(v+w) + \int_{B_{r_0}} (M-\mu)\tilde{u} + \int_{B_R \setminus B_{r_0}} (1-f)\tilde{u} \end{aligned} \tag{8}$$

But

$$\begin{aligned} \int_{B_{r_0}} (\tilde{u} - v)\mu &\leq \|\mu\|_\infty \int_{B_{r_0}} (\tilde{u} - v) \\ &\leq M \int_{B_{r_0}} (\tilde{u} - v) \end{aligned}$$

which implies that

$$\int_{B_{r_0}} (M-\mu)v \leq \int_{B_{r_0}} (M-\mu)\tilde{u}$$

On the other hand

$$\int_{B_R \setminus B_{r_0}} (1-f)v \leq \int_{B_R \setminus B_{r_0}} (1-f)\tilde{u}$$

Substituting above inequalities in (8) and then adding both sides of (7) and (8), we obtain that

$$J(v) + J_f(w) \leq J(\tilde{u}) + J_f(u_0)$$

which yields the equalities $J(v) = J(\tilde{u})$ and $J_f(w) = J_f(u_0)$, and thus $\tilde{u} = v \leq w = u_0$ by uniqueness of the minimizers.

We now proceed to show that $u \leq \tilde{u}$ in Ω. Since $\tilde{u}$ is the minimizer of J over $\mathcal{K}(B_R)$, we have

$$\int_{B_R} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \varphi + (1-\mu)\varphi \geq 0$$

for any non-negative $\varphi \in C_0^\infty(B_R)$, while u by assumption satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + (1-\mu)\varphi = 0 \quad , \forall \varphi \in C_0^\infty(\Omega)$$

Thus for non-negative $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \varphi \geq \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \qquad .$$

or $\Delta_p \tilde{u} \leq \Delta_p u$ in Ω. Since $u \mid_{\partial\Omega} = 0 \leq \tilde{u} \mid_{\partial\Omega}$, the comparison principle implies that $u \leq \tilde{u}$ in Ω as desired.

Combining the two above inequalities, we will have $u \leq \tilde{u} \leq u_0$ in Ω. Since $u_0 = 0$ in $\mathbf{R}^n \setminus B_{R_0}$, we must have $u \leq 0$ in $\Omega \setminus B_{R_0}$.

Now if $\Omega \not\subset B_{R_0}$ then, by lemma 3.1, for every $\zeta \in \partial\Omega \setminus B_{R_0}$ there are points $z \in \Omega \setminus B_{R_0}$ such that $u(z) > 0$, which contradicts the above result. Thus $\Omega \subset \Omega_0 = B_{R_0}$ which is the desired claim.□

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